

# C. HECKMAN

## SOLUTIONS

242

## TEST 2A

- (1) [30 points] For the matrix  $A$  below, find a basis for the null space of  $A$ , the row space of  $A$ , the column space of  $A$ , the rank of  $A$ , and the nullity of  $A$ . The reduced row echelon form of  $A$  is the matrix  $R$  given below.

$$A = \begin{bmatrix} 1 & -2 & 4 & 3 & 6 \\ -1 & 2 & -4 & -3 & -6 \\ -2 & 4 & -8 & 3 & 15 \end{bmatrix} \qquad R = \begin{bmatrix} 1 & -2 & 4 & 0 & -3 \\ 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

*Solution:* The null space is the set of all  $\vec{x}$  such that  $A\vec{x} = \vec{0}$ , or  $R\vec{x} = \vec{0}$ . The parameterization for these solutions is given by:

$$\begin{aligned} x_1 &= 2r - 4s + 3t \\ x_2 &= r \\ x_3 &= s \\ x_4 &= -3t \\ x_5 &= t \end{aligned} \qquad \text{so} \qquad \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = r \cdot \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + s \cdot \begin{bmatrix} -4 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t \cdot \begin{bmatrix} 3 \\ 0 \\ 0 \\ -3 \\ 1 \end{bmatrix},$$

so a basis for the null space is  $\left\{ \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -4 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ 0 \\ -3 \\ 1 \end{bmatrix} \right\}$ . Since there are three

vectors in this set, the nullity is  $\boxed{3}$ .

A basis for the column space consists of the columns of  $A$  which have a

pivot in them, namely  $\left\{ \begin{bmatrix} 1 \\ -1 \\ -2 \end{bmatrix}, \begin{bmatrix} 3 \\ -3 \\ 3 \end{bmatrix} \right\}$ . A basis for the row space of  $A$  is the

nonzero rows of  $R$ , namely  $\{[1, -2, 4, 0, -3], [0, 0, 0, 1, 3]\}$ . Since there are two vectors in each of these sets, the rank of  $A$  is  $\boxed{2}$ .

*Grading:* +10 points for the null space basis, +5 points for the nullity, the row space basis, the column space basis, and the rank. Grading for common mistakes: -3 points for choosing the columns of  $R$ ; -3 points for choosing rows from  $A$ .

(2) Let  $B = \left( \begin{bmatrix} -1 \\ 0 \\ 4 \end{bmatrix}, \begin{bmatrix} -3 \\ -1 \\ 9 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix} \right)$  and  $C = \left( \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ -2 \\ -3 \end{bmatrix}, \begin{bmatrix} -2 \\ -3 \\ -6 \end{bmatrix} \right)$  be two ordered bases for  $R^3$ .

(a) [10 points] Find the coordinates of the vector  $\vec{u} = \begin{bmatrix} -11 \\ 5 \\ 48 \end{bmatrix}$  with respect to the ordered basis  $B$ .

*Solution:* If  $\tilde{B} = \begin{bmatrix} -1 & -3 & 1 \\ 0 & -1 & 1 \\ 4 & 9 & -2 \end{bmatrix}$ , then the coordinates of  $\vec{u}$  are

$$[\vec{u}]_B = \tilde{B}^{-1} \cdot \vec{u} = \begin{bmatrix} -1 & -3 & 1 \\ 0 & -1 & 1 \\ 4 & 9 & -2 \end{bmatrix}^{-1} \cdot \begin{bmatrix} -11 \\ 5 \\ 48 \end{bmatrix} = \boxed{\begin{bmatrix} 4 \\ 6 \\ 11 \end{bmatrix}}.$$

*Grading:* +4 points for the basic formula  $[\vec{u}]_B = \tilde{B}^{-1} \cdot \vec{u}$ , +3 points for substitution, +3 points for calculation.

(b) [10 points] Find the change of basis matrix from  $C$  to  $B$ .

*Solution:* This matrix is

$$\tilde{B}^{-1} \cdot \tilde{C} = \begin{bmatrix} -1 & -3 & 1 \\ 0 & -1 & 1 \\ 4 & 9 & -2 \end{bmatrix}^{-1} \cdot \begin{bmatrix} 1 & -1 & -2 \\ 3 & -2 & -3 \\ 2 & -3 & -6 \end{bmatrix} = \boxed{\begin{bmatrix} 2 & -7 & -17 \\ 0 & 3 & 8 \\ 3 & 1 & 5 \end{bmatrix}}.$$

*Grading:* +4 points for the basic formula  $\tilde{B}^{-1} \cdot \tilde{C}$ , +3 points for substitution, +3 points for calculation. *Grading for common mistakes:* +7 points (total) for  $\tilde{C}^{-1} \cdot \tilde{B}$ ; +3 points (total) for  $\tilde{B} \cdot \tilde{C}^{-1}$  or  $\tilde{C} \cdot \tilde{B}^{-1}$ , +3 points (total) for including  $\vec{u}$  in the answer.

- (3) Let  $\vec{v}_1 = \begin{bmatrix} -1 \\ -3 \\ 3 \\ 1 \end{bmatrix}$ ,  $\vec{v}_2 = \begin{bmatrix} 3 \\ 9 \\ -9 \\ -3 \end{bmatrix}$ ,  $\vec{v}_3 = \begin{bmatrix} -3 \\ 3 \\ 3 \\ 3 \end{bmatrix}$ ,  $\vec{v}_4 = \begin{bmatrix} 11 \\ -3 \\ -15 \\ -11 \end{bmatrix}$ , and let  $W$  be the subspace spanned by these vectors.

- (a) [15 points] The set  $\{\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4\}$  is linearly dependent. Find a linearly independent set of vectors whose span is  $W$ , and the dimension of  $W$ .

*Solution:* The first part is like finding the column space of a matrix: Choose the vectors which have pivots in their columns (in the RREF).

$$\begin{bmatrix} -1 & 3 & -3 & 11 \\ -3 & 9 & 3 & -3 \\ 3 & -9 & 3 & -15 \\ 1 & -3 & 3 & -11 \end{bmatrix} \xrightarrow{RREF} \begin{bmatrix} 1 & -3 & 0 & -2 \\ 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Thus  $\{\vec{v}_1, \vec{v}_3\}$  is one possible answer. Since there are two vectors in this set, the dimension of  $W$  is  $\boxed{2}$ .

*Grading:* +5 points for the RREF, +5 points for choosing the vectors, +5 points for finding the dimension. *Grading for common mistakes:* +5 points (total) for finding a nontrivial linear combination of the vectors which adds up to  $\vec{0}$ ; +7 points (total) for finding the null space of the matrix above.

- (b) [10 points] Is  $\begin{bmatrix} 2 \\ 0 \\ 1 \\ 1 \end{bmatrix}$  in  $W$ ? Justify your answer.

*Solution:* A vector  $\vec{u}$  is in  $W$  if the system of linear equations  $[\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4 \mid \vec{u}]$  has at least one solution. We can check for this by finding the RREF:

$$\left[ \begin{array}{cccc|c} -1 & 3 & -3 & 11 & 2 \\ -3 & 9 & 3 & -3 & 0 \\ 3 & -9 & 3 & -15 & 1 \\ 1 & -3 & 3 & -11 & 1 \end{array} \right] \xrightarrow{RREF} \left[ \begin{array}{cccc|c} 1 & -3 & 0 & -2 & 0 \\ 0 & 0 & 1 & -3 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right]$$

This system has no solutions (see the third row), so  $\boxed{\text{the vector is not in } W.}$

*Grading:* +4 points for the initial augmented matrix, +3 points for the RREF, +3 points for the answer.

- (4) [25 points] Let  $A$  be the matrix  $\begin{bmatrix} -1 & -2 & 4 \\ -1 & 0 & 2 \\ -1 & -1 & 3 \end{bmatrix}$ . Find the eigenvalues of  $A$ , and for ONE of the eigenvalues, find a basis for its eigenspace.

*Solution:* To find the eigenvalues, we need to determine when  $\det(A - \lambda I) = 0$ :

$$\begin{aligned} 0 = \det(A - \lambda I) &= \begin{vmatrix} -1 - \lambda & -2 & 4 \\ -1 & -\lambda & 2 \\ -1 & -1 & 3 - \lambda \end{vmatrix} \\ &= (-1 - \lambda)(-\lambda)(3 - \lambda) + (-2)(2)(-1) + 4(-1)(-1) \\ &\quad - (-1 - \lambda)(2)(-1) - (-\lambda)(4)(-1) - (3 - \lambda)(-1)(-2) = \dots \\ &= -\lambda^3 + 2\lambda^2 - \lambda = -\lambda(\lambda^2 - 2\lambda + 1) = -\lambda(\lambda - 1)^2, \end{aligned}$$

so the eigenvalues are  $\boxed{0 \text{ and } 1}$ .

To find the eigenvectors for  $\lambda = r$ , find a basis for the null space of  $A - rI$ . For  $\lambda = 0$ ,

$$A - 0I = \begin{bmatrix} -1 & -2 & 4 \\ -1 & 0 & 2 \\ -1 & -1 & 3 \end{bmatrix} \xrightarrow{RREF} \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

and the parameterization is  $x_1 = 2s$ ,  $x_2 = s$ ,  $x_3 = s$ , so an eigenvector looks like

$$s \cdot \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} \text{ and a basis for the eigenspace is } \left\{ \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} \right\}.$$

For  $\lambda = 1$ ,

$$A - 1I = \begin{bmatrix} -2 & -2 & 4 \\ -1 & -1 & 2 \\ -1 & -1 & 2 \end{bmatrix} \xrightarrow{RREF} \begin{bmatrix} 1 & 1 & -2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

and the parameterization is  $x_1 = -s + 2t$ ,  $x_2 = s$ ,  $x_3 = t$ , so an eigenvector looks

$$\text{like } s \cdot \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + t \cdot \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} \text{ and a basis for the eigenspace is } \left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

*Grading:* +10 points for the eigenvalues, +15 points for the eigenspace basis. For the eigenvalues: +3 points for the  $\det(A - \lambda I)$  formula, +7 points for finding the roots. For the eigenspace: +4 points for  $A - \lambda I$ , +4 points for the RREF, +4 points for the parameterization, +3 points for the basis. Grading for common mistakes: -3 points for an incomplete expansion by minors (for finding the determinant), -10 points if the RREF of  $A - \lambda I$  turned out to be  $I$  (this means you made a mistake somewhere).

# C. HECKMAN

## SOLUTIONS

# 242

## TEST 2B

- (1) [30 points] For the matrix  $A$  below, find a basis for the null space of  $A$ , the row space of  $A$ , the column space of  $A$ , the rank of  $A$ , and the nullity of  $A$ . The reduced row echelon form of  $A$  is the matrix  $R$  given below.

$$A = \begin{bmatrix} 1 & 0 & 2 & 1 \\ 4 & -3 & 17 & 7 \\ 4 & -1 & 11 & 5 \end{bmatrix} \qquad R = \begin{bmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & -3 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

*Solution:* The null space is the set of all  $\vec{x}$  such that  $A\vec{x} = \vec{0}$ , or  $R\vec{x} = \vec{0}$ . The parameterization for these solutions is given by:

$$\begin{aligned} x_1 &= -2s - t \\ x_2 &= 3s + t \\ x_3 &= s \\ x_4 &= t \end{aligned} \quad \text{so} \quad \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = s \cdot \begin{bmatrix} -2 \\ 3 \\ 1 \\ 0 \end{bmatrix} + t \cdot \begin{bmatrix} -1 \\ 1 \\ 0 \\ 1 \end{bmatrix},$$

so a basis for the null space is  $\left\{ \begin{bmatrix} -2 \\ 3 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \\ 1 \end{bmatrix} \right\}$ . Since there are three vectors

in this set, the nullity is  $\boxed{2}$ .

A basis for the column space consists of the columns of  $A$  which have a pivot in them, namely  $\left\{ \begin{bmatrix} 1 \\ 4 \\ 4 \end{bmatrix}, \begin{bmatrix} 0 \\ -3 \\ -1 \end{bmatrix} \right\}$ . A basis for the row space of  $A$  is the nonzero

rows of  $R$ , namely  $\{[1, 0, 2, 1], [0, 1, -3, -1]\}$ . Since there are two vectors in each of these sets, the rank of  $A$  is  $\boxed{2}$ .

Grading: +10 points for the null space basis, +5 points for the nullity, the row space basis, the column space basis, and the rank. Grading for common mistakes: -3 points for choosing the columns of  $R$ ; -3 points for choosing rows from  $A$ .

(2) Let  $B = \left( \begin{bmatrix} -1 \\ 4 \\ -3 \end{bmatrix}, \begin{bmatrix} -3 \\ 13 \\ -8 \end{bmatrix}, \begin{bmatrix} -3 \\ 11 \\ -11 \end{bmatrix} \right)$  and  $C = \left( \begin{bmatrix} 1 \\ -2 \\ -4 \end{bmatrix}, \begin{bmatrix} 2 \\ -5 \\ -11 \end{bmatrix}, \begin{bmatrix} 4 \\ -12 \\ -27 \end{bmatrix} \right)$  be two ordered bases for  $R^3$ .

(a) [10 points] Find the coordinates of the vector  $\vec{u} = \begin{bmatrix} 18 \\ -48 \\ -106 \end{bmatrix}$  with respect to the ordered basis  $C$ .

*Solution:* If  $\tilde{C} = \begin{bmatrix} 1 & 2 & 4 \\ -2 & -5 & -12 \\ -4 & -11 & -27 \end{bmatrix}$ , then the coordinates of  $\vec{u}$  are

$$[\vec{u}]_C = \tilde{C}^{-1} \cdot \vec{u} = \begin{bmatrix} 1 & 2 & 4 \\ -2 & -5 & -12 \\ -4 & -11 & -27 \end{bmatrix}^{-1} \cdot \begin{bmatrix} 18 \\ -48 \\ -106 \end{bmatrix} = \boxed{\begin{bmatrix} 2 \\ 4 \\ 2 \end{bmatrix}}.$$

*Grading:* +4 points for the basic formula  $[\vec{u}]_C = \tilde{C}^{-1} \cdot \vec{u}$ , +3 points for substitution, +3 points for calculation.

(b) [10 points] Find the change of basis matrix from  $B$  to  $C$ .

*Solution:* This matrix is

$$\tilde{C}^{-1} \cdot \tilde{B} = \begin{bmatrix} 1 & 2 & 4 \\ -2 & -5 & -12 \\ -4 & -11 & -27 \end{bmatrix}^{-1} \cdot \begin{bmatrix} -1 & -3 & -3 \\ 4 & 13 & 11 \\ -3 & -8 & -11 \end{bmatrix} = \boxed{\begin{bmatrix} -49 & -153 & -145 \\ 50 & 157 & 147 \\ -13 & -41 & -38 \end{bmatrix}}.$$

*Grading:* +4 points for the basic formula  $\tilde{C}^{-1} \cdot \tilde{B}$ , +3 points for substitution, +3 points for calculation. *Grading for common mistakes:* +7 points (total) for  $\tilde{B}^{-1} \cdot \tilde{C}$ ; +3 points (total) for  $\tilde{B} \cdot \tilde{C}^{-1}$  or  $\tilde{C} \cdot \tilde{B}^{-1}$ , +3 points (total) for including  $\vec{u}$  in the answer.

- (3) Let  $\vec{v}_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$ ,  $\vec{v}_2 = \begin{bmatrix} -2 \\ -2 \\ 2 \\ 2 \end{bmatrix}$ ,  $\vec{v}_3 = \begin{bmatrix} 2 \\ -1 \\ 0 \\ -2 \end{bmatrix}$ ,  $\vec{v}_4 = \begin{bmatrix} 4 \\ -5 \\ 0 \\ -4 \end{bmatrix}$ , and let  $W$  be the subspace spanned by these vectors.

- (a) [15 points] The set  $\{\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4\}$  is linearly dependent. Find a linearly independent set of vectors whose span is  $W$ , and the dimension of  $W$ .

*Solution:* The first part is like finding the column space of a matrix: Choose the vectors which have pivots in their columns (in the RREF).

$$\begin{bmatrix} 0 & -2 & 2 & 4 \\ 1 & -2 & -1 & -5 \\ 0 & 2 & 0 & 0 \\ 0 & 2 & -2 & -4 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} 1 & 0 & 0 & -3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Thus  $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$  is one possible answer. Since there are three vectors in this set, the dimension of  $W$  is  $\boxed{3}$ .

*Grading:* +5 points for the RREF, +5 points for choosing the vectors, +5 points for finding the dimension. *Grading for common mistakes:* +5 points (total) for finding a nontrivial linear combination of the vectors which adds up to  $\vec{0}$ ; +7 points (total) for finding the null space of the matrix above.

- (b) [10 points] Is  $\begin{bmatrix} 2 \\ 4 \\ 4 \\ 2 \end{bmatrix}$  in  $W$ ? Justify your answer.

*Solution:* A vector  $\vec{u}$  is in  $W$  if the system of linear equations  $[\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4 \mid \vec{u}]$  has at least one solution. We can check for this by finding the RREF:

$$\left[ \begin{array}{cccc|c} 0 & -2 & 2 & 4 & 2 \\ 1 & -2 & -1 & -5 & 4 \\ 0 & 2 & 0 & 0 & 4 \\ 0 & 2 & -2 & -4 & 2 \end{array} \right] \xrightarrow{\text{RREF}} \left[ \begin{array}{cccc|c} 1 & 0 & 0 & -3 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{array} \right]$$

This system has no solutions (see the fourth row), so  $\boxed{\text{the vector is not in } W}$ .

*Grading:* +4 points for the initial augmented matrix, +3 points for the RREF, +3 points for the answer.

- (4) [25 points] Let  $A$  be the matrix  $\begin{bmatrix} 5 & -2 & -3 \\ 1 & 2 & -3 \\ 1 & -2 & 1 \end{bmatrix}$ . Find the eigenvalues of  $A$ , and for ONE of the eigenvalues, find a basis for its eigenspace.

*Solution:* To find the eigenvalues, we need to determine when  $\det(A - \lambda I) = 0$ :

$$\begin{aligned} 0 = \det(A - \lambda I) &= \begin{vmatrix} 5 - \lambda & -2 & -3 \\ 1 & 2 - \lambda & -3 \\ 1 & -2 & 1 - \lambda \end{vmatrix} \\ &= (5 - \lambda)(2 - \lambda)(1 - \lambda) + (6) + (6) + 3(2 - \lambda) - 6(5 - \lambda) + 2(1 - \lambda) = \dots \\ &= -\lambda^3 + 8\lambda^2 - 16\lambda = -\lambda(\lambda^2 - 8\lambda + 16) = -\lambda(\lambda - 4)^2, \end{aligned}$$

so the eigenvalues are  $\boxed{0 \text{ and } 4}$ .

To find the eigenvectors for  $\lambda = r$ , find a basis for the null space of  $A - rI$ . For  $\lambda = 0$ ,

$$A - 0I = \begin{bmatrix} 5 & -2 & -3 \\ 1 & 2 & -3 \\ 1 & -2 & 1 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

and the parameterization is  $x_1 = s$ ,  $x_2 = s$ ,  $x_3 = s$ , so an eigenvector looks like

$$s \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \text{ and a basis for the eigenspace is } \boxed{\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}}.$$

For  $\lambda = 4$ ,

$$A - 4I = \begin{bmatrix} 1 & -2 & -3 \\ 1 & -2 & -3 \\ 1 & -2 & -3 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} 1 & -2 & -3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

and the parameterization is  $x_1 = 2s + 3t$ ,  $x_2 = s$ ,  $x_3 = t$ , so an eigenvector looks

$$\text{like } s \cdot \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + t \cdot \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix} \text{ and a basis for the eigenspace is } \boxed{\left\{ \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix} \right\}}.$$

*Grading:* +10 points for the eigenvalues, +15 points for the eigenspace basis. For the eigenvalues: +3 points for the  $\det(A - \lambda I)$  formula, +7 points for finding the roots. For the eigenspace: +4 points for  $A - \lambda I$ , +4 points for the RREF, +4 points for the parameterization, +3 points for the basis. Grading for common mistakes: -3 points for an incomplete expansion by minors (for finding the determinant), -10 points if the RREF of  $A - \lambda I$  turned out to be  $I$  (this means you made a mistake somewhere).