MAT 271 Test #5 Solutions

(1) (20 points) Find the radius of convergence for the series $\sum_{k=5}^{\infty} \frac{(x-2)^k}{k \cdot 2^k}$. You do **not** need to find the interval of convergence.

Solution: To find the radius of convergence for a power series, you use the ratio test (or, in some cases, you can use the root test). If $a_k = \frac{(x-2)^k}{k \cdot 2^k}$, *then*

$$\frac{|a_{k+1}|}{|a_k|} = \frac{|x-2|^{k+1}}{(k+1) \cdot 2^{k+1}} \cdot \frac{k \cdot 2^k}{|x-2|^k} = \frac{|x-2|}{2} \cdot \frac{k}{k+1},$$

which converges to $\frac{|x-2|}{2}$ *as* k *approaches infinity.*

The power series will converge if the limit of the ratio of consecutive terms is less than 1 *and diverges if the limit is greater than* 1. *Thus, if* $|x-2| < 2$, *then the power series will converge for that value of* x; *if* $|x-2| > 2$, *it will diverge. Hence the radius of convergence is* 2 *(the* 2 *from the right-hand side of the equalities).*

Grading: $+5$ *points for using the ratio test;* $+5$ *points for setting up the ratio;* $+5$ *points for setting the limit of the ratio less than* 1; $+5$ *points for getting the radius of convergence.*

(2) (15 points) Consider the series $\sum_{n=3}^{\infty} \frac{4n+4}{n^2(n+2)^2}$. Find the sum of this series, using the fact that $\frac{4n+4}{n^2(n+2)^2} = \frac{1}{n^2} - \frac{1}{(n+2)^2}$.

Solution: The hint is that this series is a telescoping series. If you add up $b_n = \frac{4n+4}{n^2(n+2)^2}$ *for all* n *between* 3 *and* N, *then you get*

$$\begin{aligned}\sum_{n=3}^{N} b_n &= \left(\frac{1}{3^2} - \frac{1}{5^2}\right) + \left(\frac{1}{4^2} - \frac{1}{6^2}\right) + \left(\frac{1}{5^2} - \frac{1}{7^2}\right) + \left(\frac{1}{6^2} - \frac{1}{8^2}\right) + \cdots \\ &\quad + \left(\frac{1}{(N-2)^2} - \frac{1}{N^2}\right) + \left(\frac{1}{(N-1)^2} - \frac{1}{(N+1)^2}\right) + \left(\frac{1}{N^2} - \frac{1}{(N+2)^2}\right) \\ &= \frac{1}{3^2} + \frac{1}{4^2} - \frac{1}{(N+1)^2} - \frac{1}{(N+2)^2},\end{aligned}$$

and when you take the limit as $N \to \infty$, you see that

$$\sum_{n=3}^{\infty} b_n = \lim_{N\to\infty} \sum_{n=3}^{N} b_n = \lim_{N\to\infty} \left(\frac{1}{3^2} + \frac{1}{4^2} - \frac{1}{(N+1)^2} - \frac{1}{(N+2)^2} \right) = \frac{1}{3^2} + \frac{1}{4^2} = \frac{25}{144}.$$

Grading: +5 points for writing out the sum up to the Nth term; +5 points for checking for cancellation; +5 points for taking the limit as $N \to \infty$. Grading for common mistakes: −2 points for $1 + \frac{1}{4}$.

(3) (20 points) Find the Taylor series for $\sin(x)$ around $x = \pi$. You do **not** need to show that this series converges.

Solution: To find the Taylor series, it helps to make a table first:

k	$f^{(k)}(x)$	$f^{(k)}(\pi)$
0	$\sin x$	0
1	$\cos x$	-1
2	$-\sin x$	0
3	$-\cos x$	1
4	$\sin x$	0
5	$\cos x$	-1

The kth term of the Taylor series expansion around $x = a$ is $\frac{f^{(k)}(a)}{k!}(x-a)^k$, and the Taylor series is the sum of these terms. Using the table above, the Taylor series expansion around $x = \pi$ is

$$0 - \frac{1}{1!}(x-\pi) + 0 + \frac{1}{3!}(x-\pi)^3 + 0 - \frac{1}{5!}(x-\pi)^5 + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{(2n+1)!}(x-\pi)^{2n+1},$$

but you did not need to supply the sigma notation.

Grading: +10 points for the table (or its equivalent); +5 points for the general term of a Taylor series; +5 points for the Taylor series for $\sin(x)$. Grading for common mistakes: −3 points for bad sigma notation.

(4) (15 points each) For each of the series below, determine whether it converges. Justify your answers.

(a) $\displaystyle\sum_{k=1}^{\infty} \frac{(-1)^k \cdot (k + \ln k)}{2^k}$

Solution: When k is at least 1, $k + \ln k$ is positive, and 2^k is always positive, so the series is an alternating series. To see whether it converges, we will use the Alternating Series Test; we need to see whether the terms approach 0 and whether the **sequence** *is decreasing. Showing that the limit of the terms is 0 is not difficult (L'Hospital's Rule can be used here), but showing that the sequence is decreasing is more work than I expected, so full credit was given for showing that the terms approach 0 as $k \to \infty$.*

Grading: +5 points for deciding which test to use, +10 points for showing that the terms approach 0.

(b) $\displaystyle\sum_{k=3}^{\infty} \frac{1}{k \ln k}$

Solution: The hint was to use the Integral Test for this problem, using substitution to evaluate the limit. You need to show whether $\displaystyle\int_3^{\infty} \frac{1}{x \ln x}\, dx$ converges or diverges.

$$\int_3^{\infty} \frac{1}{x \ln x}\, dx = \int_*^* \frac{1}{xu} \cdot x\, du \qquad \begin{bmatrix} u = \ln x \\ du/dx = 1/x \\ dx = x\, du \end{bmatrix}$$

$$= \int_*^* \frac{1}{u}\, du = \ln u \Big|_*^* = \ln \ln x \Big|_3^{\infty}$$

$$= \lim_{b \to \infty} \ln \ln x \Big|_3^b = \lim_{b \to \infty} \ln \ln b - \ln \ln 3 = +\infty,$$

which shows that the integral diverges; the Integral Test implies that the series also diverges.

Grading: +5 points for setting up the integra; +5 points for evaluating the integral; +5 points for concluding that the series diverges. Grading for common mistakes: −5 points if no work was shown; −5 points for saying that $\ln \ln b - \ln \ln 3$ converges; −3 points if the integral was shown to be divergant, but you wrote down "convergant."

(5) (15 points) Determine which of the following properties the sequence $a_n = \dfrac{(-1)^n}{n}$ ($n = 1, 2, 3, \dots$) has: increasing, decreasing, bounded, convergant. You must justify your answers and also state which properties the sequence **doesn't** have.

Solution: The sequence is not increasing ($a_2 > a_3$) or decreasing ($a_1 < a_2$). It is bounded, since $|a_n| = \dfrac{1}{n} \leq 1$, and the sequence is convergent, because $\lim_{n\to\infty} a_n$ exists (and is equal to zero).

Grading: +4 points for each property (except convergant, which only had 3 points).

(6) (Extra credit, 5 points each) Evaluate the following integrals.

(a) $\displaystyle\int (\ln x)^2 \, dx$

Solution: This integral is evaluated easiest by using integration by parts twice.

$$\int (\ln x)^2 \, dx = \int 1 \cdot (\ln x)^2 \, dx = x(\ln x)^2 - \int 2 \ln x \cdot \frac{1}{x} \cdot x \, dx \qquad \begin{bmatrix} u = (\ln x)^2 & v' = 1 \\ u' = 2\ln x \cdot \dfrac{1}{x} & v = x \end{bmatrix}$$

$$= x(\ln x)^2 - 2 \int \ln x \, dx$$

$$= x(\ln x)^2 - 2 \left[x \ln x - \int \frac{1}{x} \cdot x \, dx \right] \qquad \begin{bmatrix} u = \ln x & v' = 1 \\ u' = 1/x & v = x \end{bmatrix}$$

$$= x(\ln x)^2 - 2x \ln x + 2x + C.$$

(b) $\displaystyle\int \frac{\sin^3\theta}{\cos^2\theta} \, d\theta$

Solution: This is a trigonometric integral, since it has trigonometric functions. Since the power of $\sin\theta$ is odd, you should keep one of them as a factor and convert the rest of the $\sin\theta$'s into $\cos\theta$'s and use the substitution $u = \cos\theta$.

$$\int \frac{\sin^3\theta}{\cos^2\theta} \, d\theta = \int \frac{\sin^2\theta}{\cos^2\theta} \cdot \sin\theta \, d\theta = \int \frac{1 - \cos^2\theta}{\cos^2\theta} \cdot \sin\theta \, d\theta = \int \left((\cos\theta)^{-2} - 1 \right) \cdot \sin\theta \, d\theta$$

$$= \int (u^{-2} - 1) \cdot \sin\theta \cdot \frac{du}{-\sin\theta} \qquad \begin{bmatrix} u = \cos\theta \\ du/dx = -\sin\theta \\ dx = \dfrac{du}{-\sin\theta} \end{bmatrix}$$

$$= -\left(\frac{u^{-1}}{-1} - u \right) + C = (\cos\theta)^{-1} + \cos\theta + C = \sec u + \cos u + C.$$