Rook Polynomials
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The definition of a Rook Polynomial at MathWorld[1] only applies to rectangular boards, and can be easily calculated using simple combinatorics. This document proposes to extend this definition to subsets of rectangular boards.

All information comes from [1] unless noted otherwise.

For a board B, which is a subset of a rectangular chessboard, we define the rook polynomial of B to be

$$R_B(x) = r_{B,n}x^n + r_{B,n-1}x^{n-1} + \cdots + r_{B,1}x + r_{B,0}, \tag{1}$$

where $r_{B,k}$ is the number of ways to place k rooks on B so that no rook attacks any other. It is therefore a generating function.[2]

Note that rooks are allowed to "jump over" squares which have been removed, so in Figure 1, R_1 is attacking R_2.

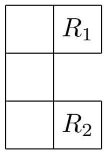

Figure 1

If B is an $m \times n$ board, then $r_{B,k} = \begin{cases} P(m,k)P(n,k)/k!, & \text{if } k \le \min(m,n) \\ 0, & \text{otherwise} \end{cases}$, where $P(n,r) = \dfrac{n!}{(n-r)!}$. This is because if we choose squares for the k-tuple $(R_1, R_2, \ldots, R_k)$, there are mn choices for R_1, $(m-1)(n-1)$ for R_2, etc. But each arrangement of rooks has been counted $k!$ times, so the number of ways to arrange k non-attacking rooks is

$$\frac{mn \cdot (m-1)(n-1) \cdot \cdots \cdot (m-k+1)(n-k+1)}{k!} = \frac{P(m,k)P(n,k)}{k!}$$

if $k \le m$ and $k \le n$.

In particular, $r_{B,0} = 1$ and $r_{B,1}$ is the number of squares in B; these two properties hold for general boards.

For non-rectangular boards, there is a recursive formula. To write this recursion in compact form, let B be a board, and s be a square of B. Then $B \backslash s$ will be the board obtained from B by deleting square s; and B/s will be the board obtained from B by deleting s as well as any squares in the same row as s, or the same column as s.

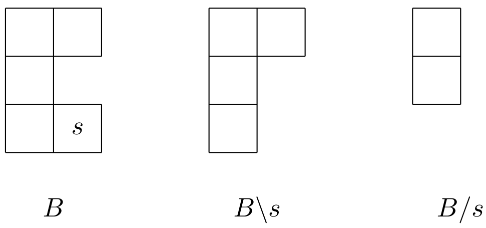

Figure 2

Then the recursion is

$$R_B(x) = R_{B\backslash s}(x) + x \cdot R_{B/s}(x), \tag{2}$$

and is proved by considering whether there is a rook on s or not.

[1] http://mathworld.wolfram.com/RookPolynomial.html
[2] http://mathworld.wolfram.com/GeneratingFunction.html

Using the example in Figure 2, B/s is a 2×1 board, so, $R_{B/s}(x) = 1+2x$. To find $R_{B\backslash s}(x)$, let t be the square in the upper right-hand corner of B/s. Then $(B\backslash s)\backslash t$ is a 3×1 board, so $R_{(B\backslash s)\backslash t}(x) = 1+3x$; and $(B\backslash s)/t$ is a 2×1 board, so $R_{(B\backslash s)/t}(x) = 1+2x$. Then

$$\begin{aligned} R_B(x) &= R_{B\backslash s}(x) + xR_{B/s}(x) \\ &= \big(R_{(B\backslash s)\backslash t}(x) + xR_{(B\backslash s)/t}(x)\big) + xR_{B/s}(x) \\ &= \big((1+3x) + x(1+2x)\big) + x(1+2x) = 1+5x+4x^2. \end{aligned}$$

If B is a sub-board of an $m\times n$ board, then the Rook Polynomial of the complement of B is given by:

$$R_{\bar{B}}(x) = \sum_{k=0}^{n}\left(\sum_{j=0}^{k}(-1)^j \cdot \frac{(m-j)!(n-j)!}{(m-k)!(n-k)!(k-j)!}\cdot r_{B,j}\right)x^k, \tag{3}$$

where $R_B(x) = \sum_{k=0}^{n} r_{B,k}x^k$. [2]

Note that rows can be swapped, and columns can be swapped, without affecting the rook polynomial. Empty rows (or columns) can be "squeezed" out, as shown in Figure 3, without affecting the rook polynomial. Basically, any operation which preserves the "is in the same row as" and "is in the same column as" relations does not affect the rook polynomial.

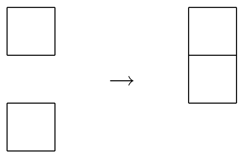

Figure 3

One last useful property is the following: If the board B has the structure shown in Figure 4, then

$$R_B(x) = R_{B_1}(x)\cdot R_{B_2}(x). \tag{4}$$

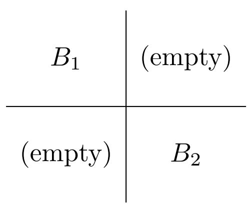

Figure 4

Polynomials for other chesspieces (which will be denoted by C) also exist; these can be calculated using the recurrence (2) by re-defining B/s. In general, B/s will be the board obtained from B by deleting s and any squares that a C on s attacks. Equation (4) will only hold if B can be partitioned into B_1 and B_2 such that if you place a C on any square of B_1, it does not attack any of the squares of B_2 (and vice versa).

The problem of the Bishop Polynomial of a board can be reduced to Rook Polynomials. Let B_B be the "black" squares of B and B_W be the "white" squares. Then rotate these boards 45 degrees clockwise to get B'_B and B'_W. Then the Bishop Polynomial of B is

$$B_B(x) = R_{B'_B}(x)\cdot R_{B'_W}(x). \tag{5}$$

For instance, if B is the board in Figure 2, then B_B, B_W, B'_B, and B'_W are displayed below.

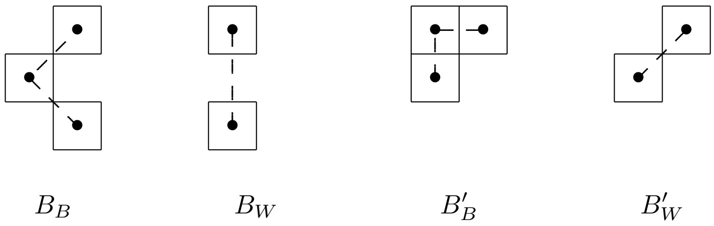

Figure 5

Thus the Bishop Polynomial of B is

$$R_B(x) = (1 + 3x + x^2) \cdot (1 + 2x + x^2) = 1 + 5x + 8x^2 + 5x^3 + x^4.$$

References:

[1] Ralph P. Grimaldi's *Discrete and Combinatorial Mathematics*, Fourth Edition, Addison-Wesley, 1999.

[2] `http://www.maths.liv.ac.uk/Past_Exams/PDF_FILES/MATH344-may05-soln.pdf` (The University of Liverpool's past examination papers, exam from MATH 344 (Combinatorics)). Hikyoo Koh also has some lecture notes from his Advanced Algorithms class (COSC5313, Fall 2006) at `http://hal.lamar.edu/~koh/5313/RookPol.htm`, but he does not give a general formula.

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