Using Graphing Calculators To Evaluate Riemann Sums

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Riemann sums are used to approximate the definite integral $\int_a^b f(x)\,dx$. Since they are difficult to evaluate by hand when n (the number of smaller intervals) is big, it would be nice to find out how to do this using your graphing calculator. Explicit instructions are given for the TI-83 (other models are similar) and the Casio 9750/9850.

Introduction

The way to approximate the definite integral $\int_a^b f(x)\,dx$ is by the *Riemann sum*

$$\sum_{k=1}^{n} f(y_i)\Delta x,$$

where the interval $[a,b]$ has been split into n smaller intervals, all with width Δx, and where y_i is in the i^{th} interval. Here is how to set up the Riemann sum for the definite integral $\int_1^3 x^2\,dx$ where $n = 10$:

(1) **Find $\Delta x = \dfrac{b-a}{n}$.** Here $\Delta x = \dfrac{3-1}{10} = 0.2$.

(2) **Find the endpoints of each interval.** The left endpoint of the first interval is the same as a. The right endpoint is $a + \Delta x$, since it (and the other intervals) are Δx units wide. The left endpoint of the second interval is the same the right endpoint of the first interval $(a + \Delta x)$, and the right endpoint is Δx more than that: $a + 2\Delta x$. In general, the left endpoint of the i^{th} interval is $a + (i-1)\Delta x$ and the right endpoint $a + i\Delta x$, where $i = 1, 2, \ldots, n$.

In our example here, the first interval is $[a, a+\Delta x] = [1, 1.2]$, the second interval is $[1.2, 1.2+0.2] = [1.2, 1.4]$, the third is $[1.4, 1.4+0.2] = [1.4, 1.6]$, and so on, so that the tenth interval is $[2.8, 2.8+0.2] = [2.8, 3]$. Note that the right endpoint of the last interval is $b = 3$, the right endpoint of the original interval, as it should be.

(3) **Determine where the function will be evaluated.** There are three main options here: the *Left-Hand Sum* (or the *Left-Hand Rule*), where we take y_i to be the left endpoint $(y_i = a + (i-1)\Delta x)$; the *Right-Hand Sum* (or the *Right-Hand Rule*), where we take $y_i = a + i\Delta x$, the right endpoint; and the *Midpoint Rule*, where we take y_i to be the midpoint of the interval, which is $y_i = a + (i-0.5)\Delta x$.[1] [2]

If we use the Left-Hand Rule with our indefinite integral, then our Riemann sum is[3]

$$\sum_{i=1}^{n} f(a+(i-1)\Delta x)\cdot\Delta x = \sum_{i=1}^{10}(1+(i-1)(0.2))^2(0.2) = (1)^2(0.2)+(1.2)^2(0.2)+(1.4)^2(0.2)+\cdots+(2.8)^2(0.2);$$

if we use the Right-Hand Rule, then the Riemann sum is

$$\sum_{i=1}^{n} f(a+i\Delta x)\cdot\Delta x = \sum_{i=1}^{10}(1+i(0.2))^2(0.2) = (1.2)^2(0.2)+(1.4)^2(0.2)+(1.6)^2(0.2)+\cdots+(3.0)^2(0.2);$$

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[1] $\frac{1}{2}\big((a+(i-1)\Delta x + a + i\Delta x)\big) = a + (i-0.5)\Delta x.$

[2] The "Sum" terminology shows up in MAT 210, the "Rule" terminology in MAT 270.

[3] The last term is not $(3)^2(0.2)$, since the sum ends at $i = n$, and we are squaring $a+(i-1)\Delta x$, not $a+i\Delta x$.

and if we use the Midpoint Rule, the Riemann sum is

$$\begin{aligned}\sum_{i=1}^{n} f(a+(i-0.5)\Delta x)\cdot\Delta x &= \sum_{i=1}^{10}(1+(i-0.5)(0.2))^2(0.2)\\ &= (1.1)^2(0.2)+(1.3)^2(0.2)+(1.5)^2(0.2)+\cdots+(2.9)^2(0.2).\end{aligned}$$

(4) **Use your calculator to evaluate the sum you've set up.** This is explained in the next two sections. Before moving on, some notation needs to be introduced. `*` will be used when you should press the $*$ button[4] on your calculator (as in `2` `+` `2` and then `EXE` or `ENTER` to add 2 and 2). If you see something in brackets [ ] after two or more keypresses, this indicates something that shows up on your screen. For instance, on the TI-83, if you press `ALPHA` `(`, a letter `K` will show up, and this will be denoted in the text by `ALPHA` `(` [*K*].

Evaluation Using a TI-83

We will be using the `sum(` and `seq(` (short for "sequence") commands to evalute the sums above. The `sum(` command adds up a list of numbers given to it, and `seq(` will produce that list. The `seq(` command needs to know what the function is ($x^2 \cdot 0.2$ here),[5] what the variable is, the starting value of the variable, the ending value of that variable, and how much the variable increases each step. Since the increase can be something other than 1, this will save us some typing; normally to evaluate the Left-Hand Sum above, you would need to enter `sum(seq((1 + (I-1)*0.2)∧2*0.2,I,1,10,1))`, but you can let `K` start at 1, end at 2.8, and increase by 0.2 each time. You would only have to put in `sum(seq(K∧2*0.2,K,1,2.8,0.2))`.

To get the `sum(` command, you need to press `2nd` `STAT` [*LIST*] `→` `→` [*MATH*] `5` [*sum(*].

To get the `seq(` command, you need to press `2nd` `STAT` [*LIST*] `→` [*OPS*] `5` [*seq(*].

The full set of keypresses for the Left-Hand Sum is: `2nd`[6] `STAT` [*LIST*] `→` `→` [*MATH*] `5` [*sum(*] `2nd` `STAT` [*LIST*] `→` [*OPS*] `5` [*seq(*] `ALPHA`[7] `(` [*K*] `∧` `2` `×` `0` `.` `2` `,` `ALPHA` `(` [*K*] `,` `1` `,` `2` `.` `8` `,` `0` `.` `2` `)` `)` `ENTER`, and you should get an answer of 7.88.

The Right-Hand Sum starts with `K` = 1.2 and goes to `K` = 3.0, increasing by 0.2 each time. So you can get the Right-Hand Sum by keying in: `2nd` `STAT` [*LIST*] `→` `→` [*MATH*] `5` [*sum(*] `2nd` `STAT` [*LIST*] `→` [*OPS*] `5` [*seq(*] `ALPHA` `(` [*K*] `∧` `2` `×` `0` `.` `2` `,` `ALPHA` `(` [*K*] `,` `1` `.` `2` `,` `3` `,` `0` `.` `2` `)` `)` `ENTER`, and you should get an answer of 9.48.

To get the Midpoint Sum, again, all you have to do is to change the limits: `K` starts at 1.1, ends at 2.9, and increases by 0.2 each time. One last time, the full set of keypresses is: `2nd` `STAT` [*LIST*] `→` `→` [*MATH*] `5` [*sum(*] `2nd` `STAT` [*LIST*] `→` [*OPS*] `5` [*seq(*] `ALPHA` `(` [*K*] `∧` `2` `×` `0` `.` `2` `,` `ALPHA` `(` [*K*] `,` `1` `.` `1` `,` `2` `.` `9` `,` `0` `.` `2` `)` `)` `ENTER`, and you should get an answer of 8.66.

Evaluation Using a Casio 9750/9850

We will be using the `Σ(` command to evalute the sums above. The `Σ(` command needs the formula ($(1+i\cdot 0.2)^2\cdot 0.2$ here),[8] the variable name (`K`), the starting value of `K`, the ending value of `K`, and the amount that `K` increases by each time (which can be omitted, and is assumed to be 1). For the Casio, `K` must start

[4] The printing is white on both models of calculators.
[5] In general, this will be $f(x)\cdot\Delta x$.
[6] This is the yellow key.
[7] This is the green key.
[8] In general, this will be $f(a+i\cdot\Delta x)\cdot\Delta x$.

at an integer,[9] so to evaluate the Left-Hand Sum above, we must enter `Σ((1 + I*0.2)∧2*0.2,I,0,9,1)`. We start at 0 to get the left endpoint of the first interval; we can add up the same thing from 1 to 10 to get the Right-Hand Sum.

To get the `Σ(` command, you need to press `OPTN` `F4` [*CALC*] `F6` `F3` [Σ(].

The full sequence of keypresses to evaluate the Left-Hand Sum is: `OPTN` `F4` [*CALC*] `F6` `F3` [Σ(] `(` `1` `+` ALPHA[10] `(` [*I*] `×` `0` `.` `2` `)` `∧` `2` `×` `0` `.` `2` `,` `ALPHA` `(` [*I*] `,` `0` `,` `9` `)` `EXE`, and you should get an answer of 7.88.

To evaluate the Right-Hand Sum, you need to change the limits: `I` starts at 1, ends at 10, and increases by 1 each time. The full sequence of keypresses to evaluate the Right-Hand Sum is: `OPTN` `F4` [*CALC*] `F6` `F3` [Σ(] `(` `1` `+` `ALPHA` `(` [*I*] `×` `0` `.` `2` `)` `∧` `2` `×` `0` `.` `2` `,` `ALPHA` `(` [*I*] `,` `1` `,` `1` `0` `)` `EXE`, and you should get an answer of 9.48.

To evaluate the Midpoint Sum, it gets messier. You need the formula for the midpoint of the i^{th} interval: $a + (i - 0.5)\Delta x = 1 + (i - 0.5)(0.2)$, and you put it in for x to get $f(a + (i - 0.5)\Delta x)$ in general, where i is now the variable. You add this up for all i's between 1 and n. For this example, this means putting in `Σ((1+(I-0.5)×0.2)∧2×0.2,I,1,10)`.

The full sequence of keypresses to evaluate the Midpoint Sum is: `OPTN` `F4` [*CALC*] `F6` `F3` [Σ(] `(` `1` `+` `(` `ALPHA` `(` [*I*] `-` `0` `.` `5` `)` `×` `0` `.` `2` `)` `∧` `2` `×` `0` `.` `2` `,` `ALPHA` `(` [*I*] `,` `1` `,` `1` `0` `)` `EXE`, and you should get an answer of 8.66.

Another example for the Midpoint Rule: Approximate $\int_2^3 \frac{1}{x}\,dx$ using $n = 100$. Then $\Delta x = \frac{3-2}{100} = 0.01$, and we need to sum up $f(a + (i - 0.5)\Delta x) \cdot \Delta x = \frac{1}{2 + (i - 0.5)(0.01)} \cdot (0.01)$, or

`1÷(2+(I-0.5)×0.01)×0.01`,

which we do by typing in `Σ(1÷(2+(I-0.5)×0.01)×0.01,I,1,100)` to get an answer of 0.4054645294. (This may take a few seconds.)

[9] Score one point for the TI's!

[10] This is the red key.