Solutions to Test 3

Problems are identified as (form, problem number, problem part). Hence, (B1a) is Form B, problem 1, part a. Form A is the light orange-colored test, Form B is the green test.

(A1a) $A \cdot \begin{bmatrix} a \\ b \end{bmatrix} = B$, **where** $A = \begin{bmatrix} 1 & -1 \\ 1 & 1 \\ 4 & 2 \\ 9 & 3 \end{bmatrix}$ **and** $B = \begin{bmatrix} -1 \\ 1 \\ 5 \\ 11 \end{bmatrix}$**.**

$$\begin{bmatrix} a \\ b \end{bmatrix} = \left(A^\top A\right)^{-1}\left(A^\top B\right) = \begin{bmatrix} 0.8076923\ldots \\ 1.1153846\ldots \end{bmatrix} = \begin{bmatrix} 21/26 \\ 29/26 \end{bmatrix}$$

$$\boxed{y = \frac{21}{26}x^2 + \frac{29}{26}x.}$$

Grading: $+3$ points for A and B, $+4$ points for the normal equation formula, $+3$ points for the equation.

(B1a) $A \cdot \begin{bmatrix} a \\ b \end{bmatrix} = B$, **where** $A = \begin{bmatrix} 1 & -1 \\ 0 & 0 \\ 4 & 2 \\ 9 & 3 \end{bmatrix}$ **and** $B = \begin{bmatrix} -5 \\ -4 \\ 4 \\ 11 \end{bmatrix}$**.**

$$\begin{bmatrix} a \\ b \end{bmatrix} = (A^\top A)^{-1}(A^\top B) = \begin{bmatrix} -0.1111\ldots \\ 3.5555\ldots \end{bmatrix} = \begin{bmatrix} -1/9 \\ 32/9 \end{bmatrix}$$

$$\boxed{y = -\frac{1}{9}\,x^2 + \frac{32}{9}\,x.}$$

Grading: +3 points for A and B, +4 points for the normal equation formula, +3 points for the equation.

(A1b) $A \cdot \begin{bmatrix} a \\ c \end{bmatrix} = B$, **where** $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 4 & 1 \\ 9 & 1 \end{bmatrix}$ **and** $B = \begin{bmatrix} -1 \\ 1 \\ 5 \\ 11 \end{bmatrix}$**.**

$$\begin{bmatrix} a \\ b \end{bmatrix} = \left(A^\top A\right)^{-1}\left(A^\top B\right) = \begin{bmatrix} 1.380116\ldots \\ -1.175438\ldots \end{bmatrix} = \begin{bmatrix} 236/171 \\ -67/57 \end{bmatrix}$$

$$\boxed{y = \frac{236}{171}\,x^2 - \frac{67}{57}.}$$

Grading: $+3$ points for A and B, $+4$ points for the normal equation formula, $+3$ points for the equation.

(B1b) $A \cdot \begin{bmatrix} a \\ c \end{bmatrix} = B$, **where** $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 4 & 1 \\ 9 & 1 \end{bmatrix}$ **and** $B = \begin{bmatrix} -5 \\ -4 \\ 4 \\ 11 \end{bmatrix}$.

$$\begin{bmatrix} a \\ b \end{bmatrix} = (A^\top A)^{-1}(A^\top B) = \begin{bmatrix} 1.81632\ldots \\ -4.85714\ldots \end{bmatrix} = \begin{bmatrix} 89/49 \\ -34/7 \end{bmatrix}$$

$$\boxed{y = \frac{89}{49}x^2 - \frac{34}{7}x.}$$

Grading: +3 points for A and B, +4 points for the normal equation formula, +3 points for the equation.

(A2) $A = \begin{bmatrix} 1 & 0 & 1 \\ -3 & 3 & -3 \\ -3 & -2 & 1 \\ -2 & 1 & -2 \\ 2 & 2 & 3 \end{bmatrix}$ **and** $B = \begin{bmatrix} 1 \\ 2 \\ 2 \\ -3 \\ 1 \end{bmatrix}$.

$$\widetilde{X} = \left(A^\top A\right)^{-1}\left(A^\top B\right) = \boxed{\begin{bmatrix} -0.3103649\ldots \\ 0.2564999\ldots \\ 0.3221406\ldots \end{bmatrix}} = \boxed{\begin{bmatrix} -2662/8577 \\ 220/8577 \\ 307/953 \end{bmatrix}}$$

Grading: $+5$ **points for** A **and** B**,** $+5$ **points for the normalization formula,** $+5$ **points for calculation.**
Partial credit: -5 **points for calculating** $A\widetilde{X}$**.**

(B2) $A = \begin{bmatrix} 2 & 1 & -3 \\ 2 & 2 & 2 \\ 3 & 3 & -2 \\ -3 & -1 & -1 \\ -2 & -2 & -1 \end{bmatrix}$ **and** $B = \begin{bmatrix} -3 \\ -3 \\ 1 \\ 3 \\ 0 \end{bmatrix}$**.**

$$\widetilde{X} = \left(A^\top A\right)^{-1}\left(A^\top B\right) = \boxed{\begin{bmatrix} -1.707529\ldots \\ 1.480398\ldots \\ -0.219041\ldots \end{bmatrix}} = \boxed{\begin{bmatrix} -2744/1607 \\ 2379/1607 \\ -352/1607 \end{bmatrix}}$$

Grading: +5 points for A and B, +5 points for the normalization formula, +5 points for calculation. Partial credit: −5 points for calculating $A\widetilde{X}$.

(A3) Gram-Schmidt.

$$\mathbf{NEW}_1 = \vec{v}_1 = \boxed{\begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \end{bmatrix}}$$

$$\mathbf{NEW}_2 = \vec{v}_2 - \frac{\mathbf{NEW}_1 \cdot \vec{v}_2}{\mathbf{NEW}_1 \cdot \mathbf{NEW}_1}\mathbf{NEW}_1$$

$$= \begin{bmatrix} 1 \\ -1 \\ 0 \\ 3 \end{bmatrix} - \frac{-4}{2}\begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \end{bmatrix} = \boxed{\begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix}}$$

(A3)

$$\mathbf{NEW}_3 = \vec{v}_3 - \frac{\mathbf{NEW}_1 \cdot \vec{v}_3}{\mathbf{NEW}_1 \cdot \mathbf{NEW}_1}\mathbf{NEW}_1 - \frac{\mathbf{NEW}_2 \cdot \vec{v}_3}{\mathbf{NEW}_2 \cdot \mathbf{NEW}_2}\mathbf{NEW}_2$$

$$= \begin{bmatrix} 3 \\ -1 \\ 1 \\ 1 \end{bmatrix} - \frac{-2}{2}\begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \end{bmatrix} - \frac{3}{3}\begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix} = \boxed{\begin{bmatrix} 2 \\ -1 \\ 1 \\ -1 \end{bmatrix}}$$

Grading: +3 points for each formula, +2 points for each calculation. Partial credit: +5 points for normalizing only.

(B3) Gram-Schmidt.

$$\mathbf{NEW}_1 = \vec{v}_1 = \boxed{\begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix}}$$

$$\mathbf{NEW}_2 = \vec{v}_2 - \frac{\mathbf{NEW}_1 \cdot \vec{v}_2}{\mathbf{NEW}_1 \cdot \mathbf{NEW}_1} \mathbf{NEW}_1$$

$$= \begin{bmatrix} 3 \\ 3 \\ -1 \\ 2 \end{bmatrix} - \frac{6}{3} \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix} = \boxed{\begin{bmatrix} 1 \\ 3 \\ 1 \\ 0 \end{bmatrix}}$$

(B3)

$$\mathbf{NEW}_3 = \vec{v}_3 - \frac{\mathbf{NEW}_1 \cdot \vec{v}_3}{\mathbf{NEW}_1 \cdot \mathbf{NEW}_1}\mathbf{NEW}_1 - \frac{\mathbf{NEW}_2 \cdot \vec{v}_3}{\mathbf{NEW}_2 \cdot \mathbf{NEW}_2}\mathbf{NEW}_2$$

$$= \begin{bmatrix} 1 \\ -4 \\ 0 \\ 2 \end{bmatrix} - \frac{3}{3}\begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix} - \frac{-11}{11}\begin{bmatrix} 1 \\ 3 \\ 1 \\ 0 \end{bmatrix} = \boxed{\begin{bmatrix} 1 \\ -1 \\ 2 \\ 1 \end{bmatrix}}$$

Grading: +3 points for each formula, +2 points for each calculation. Partial credit: +5 points for normalizing only.

(A4a)

$$\vec{v}_1 \cdot \vec{v}_2 = 1 + 0 - 1 + 0 = 0$$
$$\vec{v}_1 \cdot \vec{v}_3 = 1 + 0 - 2 + 1 = 0$$
$$\vec{v}_2 \cdot \vec{v}_3 = 1 - 3 + 2 + 0 = 0$$

Grading: $+4$ points for the dot product idea; $+2$ points for each calculation. Partial credit: Full credit for Gram-Schmidt, getting the same basis, and saying that this implies that the original set was orthogona. -1 point for only doing Gram-Schmidt.

(B4a)

$$\vec{v}_1 \cdot \vec{v}_2 = 0 + 1 + 0 - 1 = 0$$
$$\vec{v}_1 \cdot \vec{v}_3 = 0 + 1 - 2 + 1 = 0$$
$$\vec{v}_2 \cdot \vec{v}_3 = 0 + 1 + 0 - 1 = 0$$

Grading: $+4$ points for the dot product idea; $+2$ points for each calculation. Partial credit: Full credit for Gram-Schmidt, getting the same basis, and saying that this implies that the original set was orthogona. -1 point for only doing Gram-Schmidt.

(A4b)

$$\left.\begin{aligned} c_1 &= \frac{u \cdot \vec{v}_1}{\vec{v}_1 \cdot \vec{v}_1} = \frac{-6}{3} = -2 \\ c_2 &= \frac{u \cdot \vec{v}_2}{\vec{v}_2 \cdot \vec{v}_2} = \frac{0}{11} = 0 \\ c_3 &= \frac{u \cdot \vec{v}_3}{\vec{v}_3 \cdot \vec{v}_3} = \frac{-14}{7} = -2 \end{aligned}\right\} \quad \vec{p} = -2\vec{v}_1 - 0\vec{v}_2 - 2\vec{v}_3 = \boxed{\begin{bmatrix} -4 \\ 2 \\ -2 \\ -4 \end{bmatrix}}.$$

Grading: $+5$ points for the c_i formula and calculating, $+5$ points for the $\vec{p}$ formula, $+5$ points for adding. Partial credit: -8 points for not adding the vectors together, $+5$ points (total) for only finding the distance between $\vec{u}$ and the $\vec{v}_i$'s. Full credit was given for using the normal equation.

(B4b)

$$\left.\begin{aligned} c_1 &= \frac{u \cdot \vec{v}_1}{\vec{v}_1 \cdot \vec{v}_1} = \frac{6}{6} = 1 \\ c_2 &= \frac{u \cdot \vec{v}_2}{\vec{v}_2 \cdot \vec{v}_2} = \frac{-3}{3} = -1 \\ c_3 &= \frac{u \cdot \vec{v}_3}{\vec{v}_3 \cdot \vec{v}_3} = \frac{-3}{3} = -1 \end{aligned}\right\} \quad \vec{p} = \vec{v}_1 - \vec{v}_2 - \vec{v}_3 = \boxed{\begin{bmatrix} -1 \\ 1 \\ 3 \\ 1 \end{bmatrix}}.$$

Grading: $+5$ points for the c_i formula and calculating, $+5$ points for the $\vec{p}$ formula, $+5$ points for adding. Partial credit: -8 points for not adding the vectors together, $+5$ points (total) for only finding the distance between $\vec{u}$ and the $\vec{v}_i$'s. Full credit was given for using the normal equation.

(A4c) Divide each $\vec{v}_i$ by its length.

$$\boxed{\left\{ \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix}, \frac{1}{\sqrt{11}} \begin{bmatrix} 1 \\ 3 \\ 1 \\ 0 \end{bmatrix}, \frac{1}{\sqrt{7}} \begin{bmatrix} 1 \\ -1 \\ 2 \\ 1 \end{bmatrix} \right\}.}$$

Grading: +3 points for the dividing idea, +4 points for each vector. Partial credit: +5 points (total) for Gram-Schmidt; +10 points for dividing $\vec{u}$ by its length.

(B4c) Divide each $\vec{v}_i$ by its length.

$$\boxed{\left\{ \frac{1}{\sqrt{6}} \begin{bmatrix} 0 \\ -1 \\ 2 \\ 1 \end{bmatrix}, \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ -1 \\ 0 \\ 1 \end{bmatrix}, \frac{1}{\sqrt{3}} \begin{bmatrix} 0 \\ -1 \\ -1 \\ 1 \end{bmatrix} \right\}.}$$

Grading: +3 points for the dividing idea, +4 points for each vector. Partial credit: +5 points (total) for Gram-Schmidt; +10 points for dividing $\vec{u}$ by its length.