On the Tightness of the 5/14 Independence Ratio [Draft #2]

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Abstract: In 1979, Staton proved that every triangle-free graph G with maximum degree at most three has an independent set with size at least 5/14 of the number of vertices of G. Fraughnaugh (1990) and Heckman and Thomas (2001) provide shorter proofs of the same result. An analysis of the cases of equality for the main results in the last paper is presented. Also, a proof that there are only two connected triangle-free graphs with maximum degree at most three and independence ratio 5/14 is given; it is self-contained and does not require a computer search.

1. Introduction

Graph terms which are not defined in this paper use the same definitions as in [3], or any other standard graph theory textbook. We will also use $n(G)$, $e(G)$, and $\alpha(G)$ to denote, in order, the number of vertices of G, the number of edges of G, and the independence number of G, omitting the (G) when there is no danger of confusion.

A graph G is said to be *triangle-free* if no subgraph of G is isomorphic to the complete graph K_3. An *independent set* is a set of vertices, no two of which are adjacent, and the *independence number of* G is the size of a largest independent set. The *independence ratio of* G is defined to be the independence number of G divided by the number of vertices in G.

All graphs mentioned in this paper will be assumed to be simple, loopless, triangle-free, and have maximum degree at most three, unless explicitly stated otherwise.

In 1979, Staton proved:

Theorem 1.1. (Staton, [9]) *The independence ratio of a triangle-free graph with maximum degree at most three is at least* 5/14.

This settled a conjecture by Albertson, Bollobás, and Tucker [1]. A shorter proof was found later by Fraughnaugh [7] and an even shorter one by Heckman and Thomas [4]. The constant 5/14 is best possible because, as noted by Fajtlowicz [5], the generalized Petersen graph $P(7, 2)$ has 14 vertices, no triangles, and no independent set with size six.

The question remained of whether there were any other connected triangle-free graphs with maximum degree at most three with an independence ratio of exactly 5/14. One was found by Stephen Locke [8], and a computer search performed by Bajnok and Brinkmann [2] showed that these are the only two triangle-free graphs with maximum degree three having 14 vertices and independence number five. Both of these graphs are shown in Figure 5.

The question of whether there are any larger connected graphs with an independence ratio of 5/14 was settled by Fraughnaugh and Locke, who showed that the constant 5/14 is not best possible in an asymptotic sense; they showed:

Theorem 1.2. (Fraughnaugh, Locke [6]) *The independence number of every connected triangle-free graph with maximum degree at most three is at least* $\frac{11}{30}n - \frac{2}{15}$.

This bound is better than $\frac{5}{14}n$ when G has more than 14 vertices; this result, combined with Bajnok and Brinkmann's result, shows that there are no other connected graphs with an independent ratio of exactly 5/14.

This settled the problem, but since [2] uses a computer search, the result is open to question. This paper will provide a self-contained proof that there are only two connected graphs with an independence ratio of exactly 5/14, without having to do any exhaustive computer searches.

The main result from Bajnok and Brinkmann is a characterization of equality of the main result in [7], which is the following. (**M** is a particular set of graphs defined in Fraughnaugh's paper.)

Theorem 1.3. (Fraughnaugh, [7]) *If G is a triangle-free graph with maximum degree at most three, then $e \geq \frac{13}{2}n - 14\alpha$. Moreover, one of the following holds:*

(i) $e \geq \frac{13}{2}n - 14\alpha + 3$;
(ii) *G contains K_2 as a component and* $e \geq \frac{13}{2}n - 14\alpha + 2$;
(iii) *G has minimum degree equal to 2 and G contains a 4-cycle;*
(iv) *G is one of the graphs in* **M**;
(v) *G is 3-regular.*

Bajnok and Brinkmann showed that equality in Theorem 1.3 occurs for exactly three connected graphs: the two graphs with 14 vertices mentioned above, and the graph L (shown in Figure 1) consisting of eight vertices and ten edges, which has an independence number of three.

Heckman and Thomas were inspired by Fraughnaugh's proof, and looked at ways of making it more efficient. The main obstacles to a short proof were called *difficult blocks*, which turn out to be the pentagon (C_5) and the graph L below.

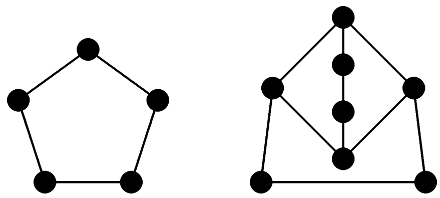

Figure 1. Difficult Blocks

A graph G is called *difficult* if, after deleting all the cut-edges of G, every component is a difficult block. The number $\lambda(G)$ is defined to be the number of components of G which are difficult. Then the following holds:

Theorem 1.4. (Heckman, Thomas, [4]) *The independence number of every triangle-free graph with maximum degree at most three is at least* $\frac{4}{7}n - \frac{1}{7}e - \frac{1}{7}\lambda(G)$.

To see why Theorem 1.4 implies the 5/14 independence ratio, consider a connected graph G. If G is difficult, then its independence ratio is at least 3/8 (and equality is attained if every block of G is L or a single edge). Otherwise,

$$\alpha(G) \geq \frac{4}{7}n - \frac{1}{7}e = \frac{5}{14}n + \frac{1}{14}(3n - 2e) = \frac{5}{14}n + \sum_{v \in V(G)} (3 - \deg v) \geq \frac{5}{14}n.$$

The quantity $d(G) = 3n - 2e$ will be called the *deficiency of G*. It is a measure of how close G is to being 3-regular; a 3-regular graph has deficiency 0.

Note that for a triangle-free graph with maximum degree at most three to have an independence ratio of exactly 5/14, it must be 3-regular and satisfy Theorem 1.4 with equality.

2. Equality in Theorem 1.4

In this section we determine all graphs for which equality holds in Theorem 1.4. We will call all such graphs *equality graphs*.

Before that, we will introduce some definitions. If X contains some of the vertices of G, we will define $\Phi(X)$ to be the number of edges with exactly one end in X. We will also use $\Phi(H)$ as an abbreviation for $\Phi(V(H))$ if H is an induced subgraph of G.

We will say that a set X of vertices is an *attachment* of G if the subgraph H of G induced by X is a difficult block (isomorphic to a pentagon or L) and $\Phi(H) = 1$. We will also say that a graph G is *obtained from H by adding attachments* if there exist graphs $G_0, G_1, \ldots, G_k$ and sets of vertices X_i for $i = 1, \ldots, k$ such that $G_0 = H$, $G_k = G$, X_i is an attachment of G_i, and $G_{i-1} = G_i \setminus X_i$. Figure 2 shows a graph obtained from Cluster (shown in Figure 6) by adding four attachments.

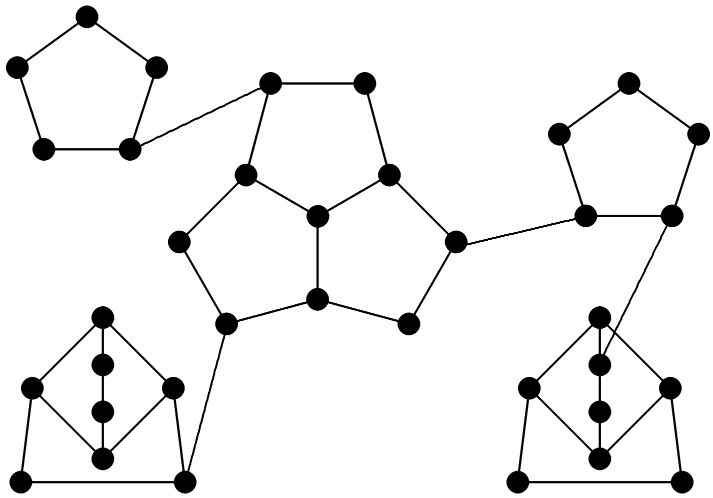

Figure 2. A Graph with Four Attachments

Next, we present a construction: Let $D_1, \ldots, D_k$ be difficult blocks, where $k \geq 2$, and select two distinct vertices u_i and v_i of D_i with degree two. The graph G with $V(G) = \bigcup_{i=1}^{k} V(D_i)$ and $E(G) = \bigcup_{i=1}^{k} E(D_i) \cup \{u_1v_2, u_2v_3, \ldots, u_{k-1}v_k, u_kv_1\}$ will be called a *ring graph*. A ring graph G with p copies of C_5 and q copies of L will be said to be an $\mathcal{R}(p, q)$ graph. (See Figure 3.)

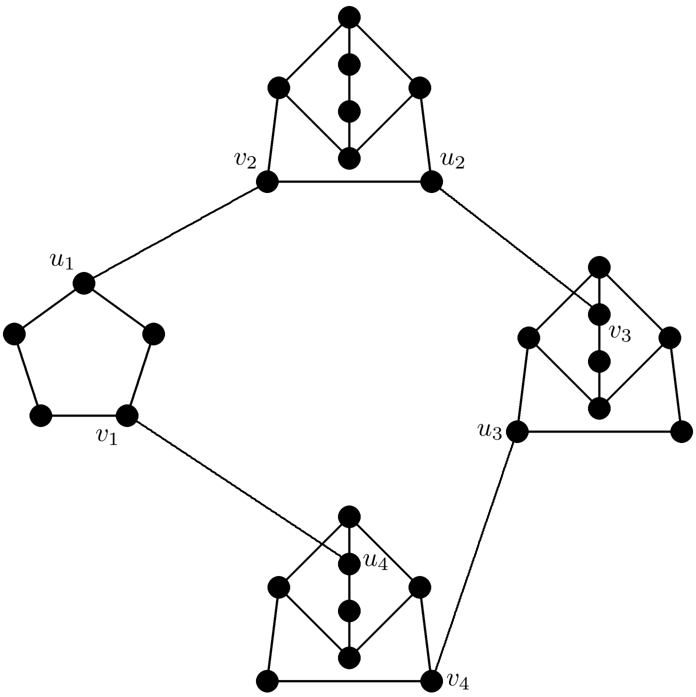

Figure 3. An $\mathcal{R}(1, 3)$ Graph

Lastly, we will let a $\mathcal{G}_2$ graph be one of the graphs depicted in Figure 4. They all have the following properties: (1) their independence number is 5; (2) each contains a unique vertex of degree two adjacent to two vertices of degree three; and (3) when the three vertices mentioned in (2) are removed, the resulting graph consists of two pentagons with an edge added between them.

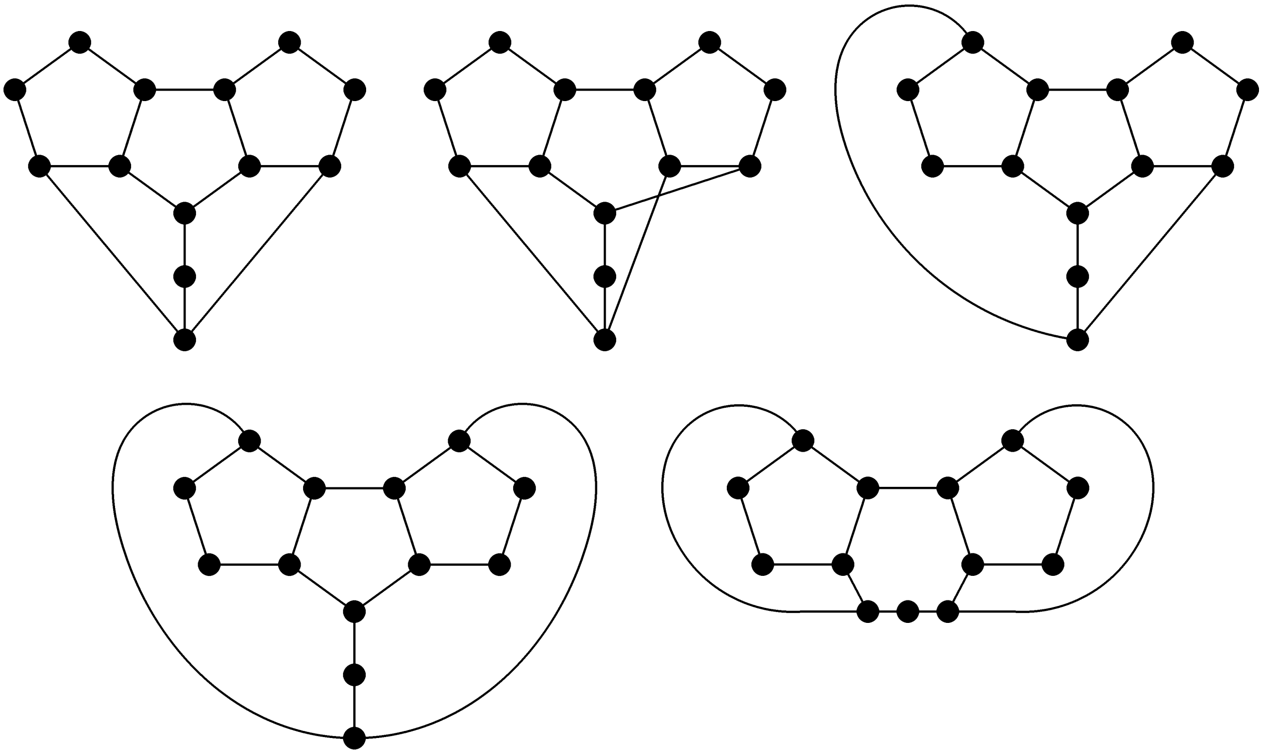

Figure 4. The $\mathcal{G}_2$ Graphs

We now present our characterization of equality graphs:

Theorem 2.1. *Suppose that G is an equality graph. Then every component of G can be obtained from a graph H by (possibly) adding attachments, such that H is:*

(i) *a difficult block;*

(ii) *a ring graph $\mathcal{R}(p, q)$;*

(iii) *a $\mathcal{G}_2$ graph;*

(iv) *one of the two 3-regular graphs F_1 and F_2 with fourteen vertices and independence number five, shown in Figure 5; or*

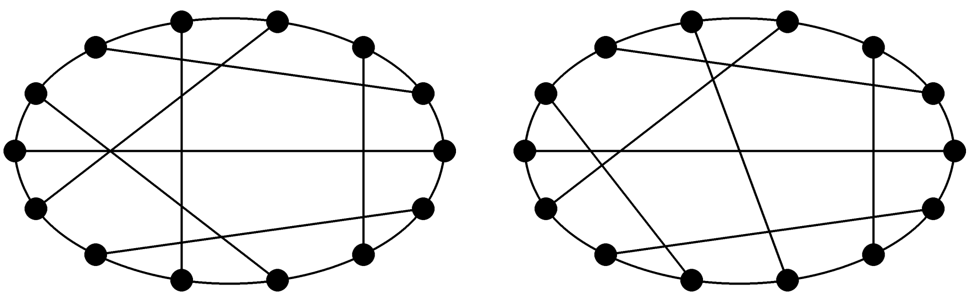

Figure 5. Two Graphs F_1 and F_2 with $n(F_i) = 14$ and $\alpha(F_i) = 5$

(v) *one of the sporadic graphs* K_2 *(the complete graph on two vertices), a heptagon (a cycle on seven vertices), the graph* $L+e$ *(*L *with an edge added between two nonadjacent vertices),* Cluster, Λ_1, Λ_2, H_1*,* H_2*, and* H_3*, depicted in Figure 6.*

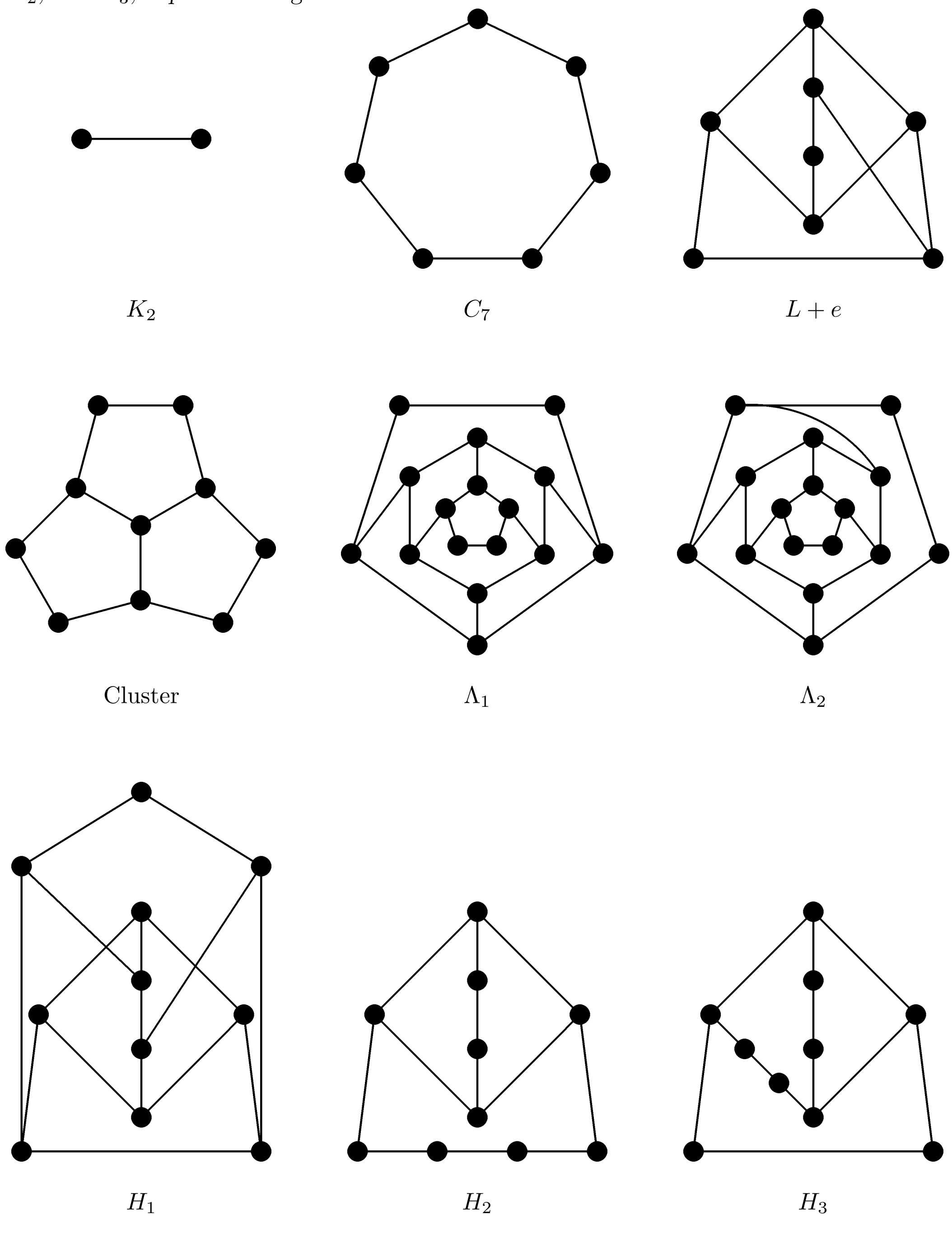

Figure 6. Graphs Mentioned in Theorem 2.1(v)

Theorem 2.1 will be proven in the next section of this paper.

Corollary. *There are exactly two connected, triangle-free graphs with maximum degree three that have an independence ratio of exactly 5/14.*

Proof: Such a graph must be an equality graph and cannot have any attachments. Since there are only two connected 3-regular equality graphs (F_1 and F_2), the result holds. □

We provide the following table of information about the graphs described in Theorem 2.1(i)–(v). Recall that the *deficiency* of a graph H is defined to be $d(H) = 3n(H) - 2e(H)$.

H	$n(H)$	$e(H)$	Girth of H	$d(H)$	$\alpha(H)$
C_5	5	5	5	5	2
L	8	10	4	4	3
$\mathcal{R}(p,q)$	$5p+8q$	$6p+11q$	4 or 5	$3p+2q$	$2p+3q$
$\mathcal{G}_2$	13	17	4 or 5	5	5
F_1 and F_2	14	21	5	0	5
K_2	2	1	∞	4	1
C_7	7	7	7	7	3
$L+e$	8	11	4	2	3
Cluster	10	12	5	6	4
H_1	11	16	4	1	4
H_2	10	12	4	6	4
H_3	10	12	5	6	4
Λ_1 and Λ_2	16	22	5	4	6

Table 1. Data for the Graphs Described in Theorem 2.1(i)–(v)

It is easy to see from Table 1 that the graphs described in Theorem 2.1(i)–(v) are equality graphs. Furthermore, we claim that if a graph G has an attachment X and $G \setminus X$ is an equality graph, then G is also an equality graph. To show this, let u be the unique vertex in X which is incident with $G \setminus X$ and v its neighbor in $G \setminus X$. Let G have n vertices and e edges. We may assume that G (and hence $G \setminus X$) is not difficult, because difficult graphs are easily seen to be equality graphs.

If the subgraph H of G induced by X is isomorphic to a pentagon, then the independence number of $G \setminus X$ is $\frac{1}{7}\Big(4(n-5) - (e-6)\Big) = \frac{1}{7}(4n-e) - 2$. Since the subgraph H has an independent set of size two disjoint from u, we see that G has an independent set with size at least $\frac{1}{7}(4n-e)$. The graph G cannot have any larger independent sets, since the independence number of $G \setminus uv$ (an upper bound on the independence number of G) is exactly $\frac{1}{7}(4n-e) - 2 + 2$. The proof where H is isomorphic to L is similar. In either case, it follows that G is an equality graph.

One final remark is the effect that an attachment X of G has on G. If $G' = G \setminus X$, then the effect of the parameters in Table 1 on G are as follows (where H is the subgraph of G induced by the vertices in X):

Parameter	H a pentagon	H isomorphic to L
$n(G)$	$n(G')+5$	$n(G')+8$
$e(G)$	$e(G')+6$	$e(G')+11$
girth (G)	$\min\{\text{girth}(G'), 5\}$	4
$d(G)$	$d(G')+3$	$d(G')+2$
$\alpha(G)$	$\alpha(G')+2$	$\alpha(G')+3$

Table 2. Data for Attachments

We thus turn our attention to proving that every equality graph satisfies Theorem 2.1. For the rest of this section, G will be a *minimal counterexample*, that is, an equality graph which violates Theorem 2.1 and has the fewest number of vertices among all such graphs.

Lemma 2.2. *Let G' be obtained from G by deleting a set X of vertices and possibly adding an edge. Let $N = |X|$, $E = e(G) - e(G')$, $\Lambda = \lambda(G') - \lambda(G)$, and suppose that every independent set I of G' can be extended to an independent set of G with size $A + |I|$. If G' is triangle-free and has maximum degree at most three, then $\Lambda \geq 7A - 4N + E$. Furthermore, if equality holds, then G' is an equality graph.*

Proof: By assumption, G' satisfies Theorem 1.1. Thus

$$\begin{aligned} 4n(G) - e(G) - \lambda(G) = 7\alpha(G) \geq 7A + 7\alpha(G') &\geq 7A + \big(4n(G') - e(G') - \lambda(G')\big) \\ &= 7A + \Big(4\big(n(G) - N\big) - \big(e(G) - E\big) - \big(\Lambda + \lambda(G)\big)\Big) \\ &= \big(4n(G) - e(G) - \lambda(G)\big) + \big(7A - 4N + E - \Lambda\big). \end{aligned}$$

Consequently $\Lambda \geq 7A - 4N + E$. If $\Lambda = 7A - 4N + E$, then all inequalities are in fact equalities, which implies that G' is an equality graph, since $7\alpha(G') = 4v(G') - e(G') - \lambda(G')$. □

Note that we will use the notation in Lemma 2.2 freely throughout the rest of the paper. In some cases, we will need to have a second set of parameters, called Y, A', N', E', and Λ', which serve the same purposes as X, A, N, E, and Λ, respectively.

Lemma 2.3. *The graph G is connected, has no attachments, and is not difficult; in particular, $\lambda(G) = 0$.*

Proof: If G is not connected, consider each component in turn. If G is difficult, then G satisfies Theorem 2.1, contrary to assumption. Lastly, if G has an attachment X, let $G' = G \setminus X$. If the subgraph of G induced by X is a pentagon, then, using the notation of Lemma 2.2, $A = 2$, $N = 5$, and $E = 6$. This implies that $\lambda(G') = \Lambda \geq 0$. Since G is connected and not difficult, $\lambda(G') = 0$, and Lemma 2.2 further implies that G' is an equality graph and is thus of the form (i)–(v). But this means that G' is obtained from one of the graphs in (i)–(v) by adding at least one attachment, and hence so is G, which then satisfies Theorem 2.1. A similar set of contradictions occurs if the subgraph induced by X is isomorphic to L. □

Recall that $\Phi(X)$ is defined to be the number of edges of G with exactly one end in X, and that we will abuse notation by letting $\Phi(H)$ be $\Phi(V(H))$ when H is an induced subgraph of G.

Lemma 2.4. *If H is an induced subgraph of G which is a pentagon, then $\Phi(H) \geq 3$. If H is an induced subgraph of G isomorphic to L, then $\Phi(H) = 4$.*

Proof: Suppose that H is an induced subgraph of G which is a difficult block, and that $\Phi(H) \leq 2$; we will then let $X = V(H)$ and $G' = G \setminus X$. If H is a pentagon, then $A = 2$, $N = 5$, and $E = 5 + \Phi(H)$. If $\Phi(H) = 0$, then G is a difficult block, contrary to Lemma 2.3; if $\Phi(H) = 1$, then G has an attachment, also violating Lemma 2.3. Lemma 2.2 then implies that $\lambda(G') = \Lambda \geq \Phi(H) - 1 \geq 1$. We have equality here, because otherwise G would have an attachment or a difficult component. Also, G' has no other components, for the same reason. Furthermore, the graph obtained from G' by contracting every difficult block to a single vertex is a path (possibly with only one vertex), because otherwise G would have an attachment. Further, H must be adjacent to a vertex in each end-block of G' (adjacent to two vertices in G' if G' is a difficult block); but then G is a ring graph, which satisfies Theorem 2.1(ii).

The proof where H is isomorphic to L is similar, except that for every vertex v of degree two, L has an independent set with size three which is disjoint from all its vertices of degree two except v. This fact strengthens $\Phi(H) \geq 3$ to $\Phi(H) \geq 4$, which implies equality, because $\Phi(H) \leq 4$. □

For a difficult graph D, let $b(D)$ be the number of difficult blocks of D. Then Lemma 2.4 implies the following:

Lemma 2.5. *Suppose that D is a difficult graph that is an induced subgraph of G. Then $\Phi(D) \geq 2\lambda(D) + b(D) \geq 3\lambda(D)$. Furthermore, if equality holds throughout, then every component of D is a pentagon.*

Proof: Let G and D be as stated and let $\mathcal{B}$ be the set of all difficult blocks B of D; then $|\mathcal{B}| = b(D)$. For $B \in \mathcal{B}$ we define $\mathrm{val}(B)$ to be the number of cut-edges of D with one end in $V(B)$ (and hence precisely one end in $V(B)$), and we define $\xi(B)$ to be the number of edges of G with one end in $V(B)$ and the other in $V(G) \setminus V(D)$. Then for every $B \in \mathcal{B}$ we have $\xi(B) + \mathrm{val}(B) = \Phi(B)$, but $\Phi(B) \geq 3$ by Lemma 2.4. By summing over all $B \in \mathcal{B}$ we obtain $\Phi(D) + 2\big(b(D) - \lambda(D)\big) \geq 3b(D)$, which gives the desired result.

If $\Phi(D) = 3\lambda(D)$, then every component C of D has $b(C) = 1$. Lemma 2.4 then implies that C must be a pentagon. □

Note that if $G' = G \setminus X$, Lemma 2.2 provides a lower bound on the number of difficult components of G' and that Lemma 2.5 provides an upper bound on the same quantity: In that case, we will let D be the difficult components of G' and then deduce that $\lambda(G') \leq \frac{1}{3}\Phi(D) \leq \frac{1}{3}\Phi(X)$.

We will now show that G has minimum degree at least two, how vertices of degree two appear in G, and that the minimum degree of G is exactly two.

Lemma 2.6. *The graph G has minimum degree at least two.*

Proof: Suppose otherwise. If G has a vertex of degree zero, then G consists of one vertex and no edges; but then G is not an equality graph. Hence G has a vertex u of degree one; let v be its neighbor, and let d_v be $\deg v$. If $d_v = 1$, then G is isomorphic to K_2, and G satisfies Theorem 2.1(v). Otherwise, we delete $X = \{u, v\}$ from G to obtain G'; then $A = 1$, $N = 2$, and $E = d_v$, and Lemma 2.2 and Lemma 2.5 imply that $\frac{2}{3} \geq \frac{1}{3}d_v \geq \lambda(G') \geq d_v - 1 \geq 1$, a contradiction. We thus conclude that G has no vertices with degree less than two. □

Lemma 2.7. *If G has two adjacent vertices u and v of degree two, they are in the vertex-set of some pentagon in G. Furthermore, all vertices in this pentagon, other than u and v, have degree three, and G is not 2-regular.*

Proof: Suppose that such vertices u and v exist. Let t (resp. w) be the other neighbor of u (resp. v), and let $G' = G \setminus \{u, v\} \cup \{tw\}$. If G' is triangle-free, then $A = 1$, $N = 2$, and $E = 2$, which implies that $\lambda(G') \geq 1$. Since G' is connected, G' must be a difficult component, and if G' is not a difficult block, then G has an attachment. Thus G' is isomorphic to a pentagon (which means that G is isomorphic to C_7) or L (which means that G is isomorphic to H_2 or H_3). But in any case, G satisfies Theorem 2.1(v).

We conclude that G' must contain a triangle, which uses the edge tw. Thus there is a vertex x of G' which is adjacent to t and w. Then G contains the 5-cycle $tuvwxt$, which is the vertex-set of a pentagon P in G. Since a pentagon is a difficult block, we must have $\Phi(P) \geq 3$, which implies that t, w, and x all have degree three, as desired. □

Lemma 2.8. *The graph G has no vertices of degree two adjacent to two vertices of degree three.*

Proof: Suppose otherwise, and let u be such a vertex, which is adjacent to the vertices v and w. Let $X = \{u, v, w\}$ and $G' = G \setminus X$, so that $A = 1$, $N = 3$, $E = 6$, and $\Phi(X) = 4$. Lemma 2.2 and Lemma 2.5 then imply that $\frac{4}{3} \geq \lambda(G') \geq 1$, so that $\lambda(G') = 1$.

Let D be the difficult component of G'; by Lemma 2.5, D must be a pentagon, L, or two pentagons with an edge between them. If D is isomorphic to L, then G is isomorphic to H_1, because G is triangle-free and Lemma 2.5 implies that every edge in $\delta(X)$ is incident with D. But then G satisfies Theorem 2.1(v). If D consists of two pentagons and an edge between them, then either $\alpha(G) = 5$ and G is a $\mathcal{G}_2$ graph (and satisfies Theorem 2.1(iii)), or G has 13 vertices, 17 edges, and an independent set with size six (which implies that G is not an equality graph).

Thus D is a pentagon. If all four edges in $\delta(X)$ are incident with D, then, because G is triangle-free, G is isomorphic to $L{+}e$ and satisfies Theorem 2.1(v). Consequently, only three edges of $\delta(X)$ are incident with D, and the fourth is incident with a nondifficult component. By symmetry, we may assume that both neighbors v_1 and v_2 of v and one neighbor w_1 of w other than u are in $V(D)$, with v_1 and v_2 being nonadjacent.

If w_1 is adjacent to v_1 and v_2, then the subgraph H of G induced by the set $Y = \{u, v, w\} \cup V(D)$ is L, and $\Phi(H) = 1$, contradicting Lemma 2.4. So w_1 is only adjacent to one of v_1 and v_2. Letting I consist of the vertices of degree two in D and v and deleting the set Y above yields $A' = 3$, $N' = 7$, and $E' = 10$. Then $\lambda(G \setminus Y) \geq 3$ by Lemma 2.2, but $G \setminus Y$ is connected. Consequently, D is not a pentagon, and we cannot have $\lambda(G') = 1$ after all. This contradiction shows that no such vertices as u, v, and w above can exist. □

Lemma 2.9. *The graph G is not 3-regular.*

Proof: Suppose G is 3-regular. We will let v be a vertex of G, and if G contains a 4-cycle, we will choose v to lie in one. We will further let the neighbors of v be u_1, u_2, and u_3, $X = \{v, u_1, u_2, u_3\}$ and $G' = G \setminus X$. We have $A = 1$, $N = 4$, and $E = 9$, since G is 3-regular. Lemma 2.2 implies that $\lambda(G') \geq 0$. Furthermore, the deficiency of G' is easily seen to be exactly six. Since every vertex in every difficult component C of G' has degree three in G, and every difficult component has at least four vertices of degree two, we must have $\Phi(C) \geq 4$. Thus $\lambda(G') \leq 1$ as well.

Suppose that $\lambda(G') = 1$, and let D be the difficult component of G'. Then D must have at most six vertices of degree two; hence D must be a pentagon, L, or two copies of L with an edge between them. If D is the third option, then G has a 4-cycle, but v is not contained in a 4-cycle, contrary to how v was chosen. If D is a pentagon, then two of the neighbors of v (denoted u_1 and u_2) are each adjacent to two nonadjacent vertices of D, which means that G has a 4-cycle; however, since v cannot lie in a 4-cycle, this contradicts the choice of v. Lastly, if D is isomorphic to L, then two of the neighbors of v (again denoted u_1 and u_2) have a common neighbor w not in $V(D)$, and the other four edges are incident with D. There is an independent set I' of G with size five containing u_1, u_2, and three vertices of D; if we let $Y = I' \cup N(I')$, we have $A' = 5$, $N' = 13$, $E' = 20$, and $\Phi(Y) = 1$, which implies that $\frac{1}{3} \geq \lambda(G \setminus Y) \geq 3$. This shows that $\lambda(G') = 0$.

Lemma 2.2 thus implies that G' is an equality graph, and having fewer vertices than G, G' satisfies Theorem 2.1. If G contains a 4-cycle vu_1wu_2v, then w has degree at most one in G'. The vertex w cannot have degree zero in G', because K_1 would be a component of G' which is not an equality graph. Letting w' be the neighbor of w in $G \setminus X$, we deduce that w' cannot have degree one, because then one of the neighbors of v will have to be adjacent to w and w', creating a triangle in G. Hence G' must consist of K_2 (vertices w and w') and at least one attachment. If G' has at least two difficult end-blocks, then $\Phi(G') \geq 4+4+2 = 10$; but there are only 6 edges in $\Phi(G')$. Thus G' has one end-block D, with D a pentagon. But then G' has a deficiency of at least $2+1+4 = 7$; this contradicts the fact that $\Phi(G') = 6$.

Therefore G' must have exactly six vertices of degree two and girth at least five. Thus G' cannot contain L, $L+e$, H_1, or H_2; and G' cannot contain F_1 or F_2, since this graph would be a component of G. Also, we can rule out C_7, since it has a deficiency of 7, and adding attachments only increases the deficiency.

Since G has girth five, any attachment of G' must be a pentagon. If G' has a pentagon D as an attachment, then $G' \setminus D$ is also an equality graph; but $\Phi(G' \setminus D) = 3$, which is impossible since none of the remaining graphs in Table 1 has a deficiency of three. If G' is not connected, then one of its components would have to have deficiency at most three; thus, this cannot happen, either.

Therefore, G' is either Cluster, H_3, or an $\mathcal{R}(2,0)$ graph (the only graphs remaining in Table 1 with a deficiency of 6). G' cannot be Cluster, because Cluster has three pairwise nonadjacent vertices of degree three. If G' were Cluster, then G would have 14 vertices, 21 edges, and an independent set with size six, and G would not actually be an equality graph.

But G' cannot be either of the remaining graphs; in each case, either G is a graph with 14 vertices, 21 edges, and independence number six, or G is F_1 or F_2. □

Lemma 2.10. *The graph G is Λ_0 or Λ_1.*

Proof: By Lemma 2.6, Lemma 2.7 and Lemma 2.9, there is a vertex v_1 of degree two adjacent to a vertex v_5 of degree three. By Lemma 2.8, v_1's other neighbor v_2 has degree two as well. Lemma 2.7 also implies that the other neighbor v_3 of v_2 has a common neighbor with v_5, which will be denoted v_4, and that v_3, v_4, and v_5 all have degree three.

We will let v_{3+i} be the neighbor of v_i not in $X = \{v_1, \ldots, v_5\}$, for $i = 3, 4, 5$. Then v_6, v_7, and v_8 are pairwise distinct unless $v_6 = v_8$. If so, then $I = \{v_2, v_4, v_6\}$ is an independent set of G; let $X = I \cup N(I)$ and $G' = G \setminus X$. Let u be the neighbor of v_6 other than v_3 and v_5. If $u = v_7$, then this vertex has degree three by Lemma 2.8, and we have $A = 3$, $N = 7$, $E = 10$, and $\Phi(X) = 1$, so $0 = \lambda(G') \geq 3$; this contradiction proves that $u \neq v_7$.

Now, uv_7 is not an edge of G, because then the subgraph H of G induced by $\{v_1, \ldots, v_7, u\}$ would have $\Phi(H) \leq 2$, which violates Lemma 2.4, as H is isomorphic to L. Furthermore, Lemma 2.7 implies that u and v_7 have degree three, so that $A = 3$, $N = 8$, $E = 13$, and $\Phi(X) = 4$, so $\frac{4}{3} \geq \lambda(G') \geq 2$, another contradiction.

Hence $v_6 \neq v_8$. We will now delete the set $X = \{v_1, \ldots, v_5\}$ and add the edge e_i to obtain G_i, where $e_1 = v_6v_8$, $e_2 = v_6v_7$, and $e_3 = v_7v_8$.

Note that e_i is not an edge of G, for any i; otherwise, we could delete X to obtain a graph G', and we can extend any independent set I of G' by adding two vertices of X. Which vertices we can add depends on which end of e_i is in I; in any case, we have $A = 2$, $N = 5$, and $E = 8$. Lemma 2.2 implies that $\Lambda \geq 2$; however, the difficult component of G' not containing e_i is an attachment of G, contrary to Lemma 2.3. Thus $e_i \notin E(G)$.

Now we show that v_6, v_7, and v_8 all have degree three. If v_6 has degree two, then by Lemma 2.7, its neighbor u has degree two as well. Now let $I = \{v_2, v_6\}$, $X = I \cup N(I)$, and $G' = G \setminus X$; then $A = 2$, $N = 5$, $E = 7$, and $\Phi(X) = 3$. Thus $\frac{3}{3} \geq \Lambda \geq 1$. Since $\Phi(X) = 3$, G' must consist of a single pentagon, and consequently G is isomorphic to Cluster, which contradicts Theorem 2.1(v).

Now suppose that v_7 has degree two. Once again, its neighbor u must have degree two as well, and the other neighbor of u must be v_6 (by symmetry). Now let $I = \{v_2, v_5, v_7\}$, $X = I \cup N(I)$, and $G' = G \setminus X$. Note that $\Phi(X) = 4$, but since two edges are incident with a vertex of degree one (in G'), only two edges can be incident with any difficult components of G'. Lemma 2.2 then implies that $\Lambda \geq 3$, which cannot happen. Thus v_6, v_7, and v_8 all have degree three, as desired.

We now claim the following:

(1) *The graph G_i contains a triangle, for all $i = 1$, 2, 3.*

To show (1), let $w_1 = v_7$, $w_2 = v_8$, and $w_3 = v_6$, and assume that G_i is triangle-free for some i. We then have $A = 2$, $N = 5$, and $E = 8 - 1$; Lemma 2.2 then implies that $\lambda(G_i) \geq 1$. If G_i had two difficult components, the one without the edge e_i would provide an attachment of G. Thus $\lambda(G_i) = 1$, and so G_i contains a difficult component D which contains e_i. Furthermore, D is a difficult block; otherwise an end-block B of D which does not have e_i would have $\Phi(B) \leq 2$ in G, contrary to Lemma 2.4.

If $D = G_i$, then D cannot be isomorphic to L, since G would contain an odd number of vertices of degree two (w_i has degree two in G' and three in G) and violate Lemma 2.7; thus D is a pentagon. In order for G to satisfy Lemma 2.7, G must be isomorphic to Cluster or H_3 and thus satisfy Theorem 2.1(v).

Now suppose that G_i contains a nondifficult component. We deduce that D is isomorphic to L, because otherwise G would violate Lemma 2.7. If $i = 1$, then we can find an independent set I' of G contained in $V(D) \cup X \setminus \{v_3, v_4, v_5\}$ with size five. Since $Y = I' \cup N(I') = V(D) \cup X \setminus \{v_4\}$, we have $A' = 5$, $N' = 12$, and $E' = 16$, if we delete the set Y from G. Then $G \setminus Y$ is connected; however, Lemma 2.2 implies that $\lambda(G_i) \geq 3$. If $i \neq 1$, we can find independent sets which produce a similar contradiction. This proves (1).

Claim (1) implies that there are vertices v_9 adjacent to v_6 and v_8, v_{10} adjacent to v_6 and v_7, and v_{11} adjacent to v_7 and v_8. The next step is showing that these vertices are distinct.

(2) *There is no vertex u adjacent to v_6, v_7, and v_8.*

To prove (2), we will assume that such a vertex u exists and show that this leads to a contradiction. In that case, we let v, w, and x be the neighbors of v_6, v_7, and v_8 not in $\{v_1, \ldots, v_8\}$, respectively. The vertices v, w, and x exist, since v_6, v_7, and v_8 all have degree three by Lemma 2.8. Suppose that v, w, and x are not all distinct vertices.

If $v = w = x$, then G has ten vertices, 14 edges, and an independent set of size four, which implies that G is not an equality graph.

If $v = w$, then we let $I' = \{v_2, v_5, v_6, v_7\}$, and if $v = x$, then we let $I' = \{v_2, v_4, v_6, v_8\}$. In either case, letting $Y = I' \cup N(I')$, we have $A = 4$, $N = 10$, $E = 15$, and $\Phi(Y) \leq 2$, which implies that $\frac{2}{3} \geq \lambda(G \setminus Y) \geq 3$. Hence v, w, and x are pairwise distinct.

If v has degree two, then by Lemma 2.7, v is adjacent to another vertex of degree two, and these vertices are in the vertex-set of some pentagon. But this implies that v is adjacent to w or x, and that vertex has degree two as well. If v is adjacent to w, we let $I' = \{v_1, v_3, v_7, v\}$; if v is adjacent to x, then we let $I' = \{v_1, v_3, v_8, v\}$. In either case, deleting $Y = I' \cup N(I')$, we have $A' = 4$, $N' = 10$, and $E' = 14$; but $G \setminus Y$ is connected, so $1 \geq \lambda(G \setminus Y) \geq 2$, the latter following by Lemma 2.2. This shows that v, and hence x, has degree three. It follows that w has degree three, because otherwise v or x would also have degree two.

If vw is not an edge of G, we will let $I' = \{v_2, v_5, v_6, v_7\}$ and $Y = I' \cup N(I')$; then $A' = 4$, $N' = 11$, $E' = 18$, and $\Phi(Y) = 5$. Lemma 2.2 and Lemma 2.5 imply that $\frac{5}{3} \geq \lambda(G \setminus Y) \geq 2$. Hence we may assume that vw and wx are edges of G, the latter by symmetry. Using the same Y, we now have $E' = 17$ and $\Phi(Y) = 2$; thus $\frac{2}{3} \geq \lambda(G \setminus Y) \geq 1$. This proves (2).

Claim (2) implies that v_9, v_{10}, and v_{11} are pairwise distinct. Each of v_9, v_{10}, and v_{11} is adjacent to two vertices of degree three and hence has degree three by Lemma 2.8.

Finally, let $I = \{v_2, v_3, v_7, v_8\}$ and delete $X = \{v_1, \ldots, v_{11}\}$ to obtain the graph G'. Since $A = 4$, $N = 11$, $E = 17$, and $\Phi(X) = 3$, $\frac{3}{3} \geq \Lambda \geq 1$. Thus G' is a pentagon; otherwise some previous lemma is violated. The edges between X and this pentagon have to connect so that Lemma 2.8 holds, which means G is Λ_1 or Λ_2. □

Lemma 2.10 contradicts Theorem 2.1(v). This finishes the proof of Theorem 2.1.

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4. References

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