

242 Test 2 A

SOLUTIONS

1. [30 points] For the matrix A below, find a basis for the null space of A , a basis for the row space of A , a basis for the column space of A , the rank of A , and the nullity of A . The reduced row echelon form of A is the matrix R given below.

$$A = \begin{bmatrix} -4 & -4 & -8 \\ 4 & 4 & 8 \\ 4 & 4 & 8 \end{bmatrix} \qquad R = \begin{bmatrix} 1 & 1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Solution: To find a basis for the null space of A , you need to solve the equation $A\vec{x} = \vec{0}$ and parameterize the solutions. Looking at the reduced row echelon form, it is easy to see that $x_2 = r$ and $x_3 = s$ are free variables; then $x_1 = -r - 2s$, so a vector in the null space is of the form

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = r \cdot \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + s \cdot \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix},$$

and $\left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} \right\}$ is a basis for the null space. The number of vectors in this basis is 2, so the nullity of A is $\boxed{2}$.

To find a basis for the row space of A , take the nonzero rows of R :

$$\boxed{\{[1, 1, 2]\}}.$$

To find a basis for the column space of A , take the columns of A which have a pivot in their column in R , namely

$$\boxed{\left\{ \begin{bmatrix} -4 \\ 4 \\ 4 \end{bmatrix} \right\}}.$$

The number of vectors in either of these sets is the rank of A ; thus $\text{rank } A = \boxed{1}$.

Grading: +5 points for the row space basis, +5 points for the column space basis, +5 points for the nullity, +5 points for the rank, and +10 points for the null space basis. Grading for common mistakes: -2 points for choosing rows from A ; -3 points for choosing columns from R ; -3 points for the rank or nullity not equalling the number of vectors in the appropriate basis.

2. Let B be the (ordered) basis $\left(\begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -2 \\ -3 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ -3 \end{bmatrix}\right)$ and C the basis $\left(\begin{bmatrix} 1 \\ -3 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ -10 \\ 5 \end{bmatrix}, \begin{bmatrix} -3 \\ 12 \\ -2 \end{bmatrix}\right)$.

a. [15 points] Find the coordinates of $\begin{bmatrix} -2 \\ 13 \\ 6 \end{bmatrix}$ with respect to the basis C .

Solution: Letting $\vec{u} = \begin{bmatrix} -2 \\ 13 \\ 6 \end{bmatrix}$,

$$[\vec{u}]_C = \tilde{C}^{-1} \cdot \vec{u} = \begin{bmatrix} 1 & 3 & -3 \\ -3 & -10 & 12 \\ 2 & 5 & -2 \end{bmatrix}^{-1} \cdot \begin{bmatrix} -2 \\ 13 \\ 6 \end{bmatrix} = \boxed{\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}}.$$

Grading: +5 points for the formula, +5 points for substitution, +5 points for calculation.

b. [15 points] If the coordinates of \vec{u} with respect to C are $\begin{bmatrix} 3 \\ 13 \\ 12 \end{bmatrix}$, what are the coordinates of \vec{u} with respect to B ?

Solution:

$$[\vec{u}]_B = \tilde{B}^{-1} \cdot \tilde{C} \cdot [\vec{u}]_C = \begin{bmatrix} -1 & 1 & 0 \\ 1 & -2 & -1 \\ 1 & -3 & -3 \end{bmatrix}^{-1} \cdot \begin{bmatrix} 1 & 3 & -3 \\ -3 & -10 & 12 \\ 2 & 5 & -2 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ 13 \\ 12 \end{bmatrix} = \boxed{\begin{bmatrix} 14 \\ 20 \\ -31 \end{bmatrix}}.$$

Grading: +5 points for the formula, +5 points for substitution, +5 points for calculation.

Grading for common mistakes: +10 points (total) for $\begin{bmatrix} 517 \\ -256 \\ -87 \end{bmatrix}$ (backwards); +7 points (total) for

$\begin{bmatrix} -186 \\ 642 \\ -271 \end{bmatrix} = CB^{-1}[u]$; +7 points (total) for $\begin{bmatrix} -235 \\ 321 \\ 423 \end{bmatrix} = BC^{-1}[u]$; +7 points (total) for $B^{-1}[u]$ or $C^{-1}[u]$.

3. Let $\vec{v}_1 = \begin{bmatrix} 0 \\ -4 \\ 1 \\ 2 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} -1 \\ -1 \\ -5 \\ 1 \end{bmatrix}$, $\vec{v}_3 = \begin{bmatrix} -5 \\ 0 \\ -3 \\ 4 \end{bmatrix}$, and $\vec{v}_4 = \begin{bmatrix} 15 \\ 15 \\ -18 \\ -21 \end{bmatrix}$.

- a. [15 points] The set $\{\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4\}$ is linearly dependent. Find a nontrivial linear combination of these vectors that adds up to $\vec{0}$.

Solution: To do this, you need to find a nontrivial solution to $c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3 + c_4\vec{v}_4 = \vec{0}$, whose matrix looks like $[\vec{v}_1 \ \vec{v}_2 \ \vec{v}_3 \ \vec{v}_4 \mid \vec{0}]$, or

$$\left[\begin{array}{cccc|c} 0 & -1 & -5 & 15 & 0 \\ -4 & -1 & 0 & 15 & 0 \\ 1 & -5 & -3 & -18 & 0 \\ 2 & 1 & 4 & -21 & 0 \end{array} \right] \xrightarrow{\text{RREF}} \left[\begin{array}{cccc|c} 1 & 0 & 0 & -5 & 0 \\ 0 & 1 & 0 & 5 & 0 \\ 0 & 0 & 1 & -4 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right].$$

The solutions to this system are parameterized by

$$\begin{aligned} c_1 &= 5\alpha \\ c_2 &= -5\alpha \\ c_3 &= 4\alpha \\ c_4 &= \alpha \end{aligned}$$

where α can be any real number. Choosing $\alpha = 1$ produces the linear combination

$$\boxed{5\vec{v}_1 - 5\vec{v}_2 + 4\vec{v}_3 + \vec{v}_4 = \vec{0}}$$

(but choosing any other nonzero real value for α will also produce a correct answer).

Grading: +5 points for setting up the system of linear equations; +5 points for parameterizing it; +5 points for finding a linear combination. Grading for common mistakes: +7 points (total) for using the “column space” method instead.

- b. [15 points] Find a basis for the subspace spanned by $\{\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4\}$, and the dimension of that subspace.

Solution: To find a basis, choose the vectors which have a pivot in their columns in the reduced

row echelon form. This basis is $\boxed{\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}}$ or $\left\{ \begin{bmatrix} 0 \\ -4 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ -1 \\ -5 \\ 1 \end{bmatrix}, \begin{bmatrix} -5 \\ 0 \\ -3 \\ 4 \end{bmatrix} \right\}$. The dimension of the

subspace is the number of vectors in the basis, namely $\boxed{3}$.

Grading: +10 points for the basis, +5 points for the dimension. Grading for common mistakes: +7 points (total) for using the “null space” method instead.

c. [10 points] Is the vector $\begin{bmatrix} 6 \\ 5 \\ 7 \\ -7 \end{bmatrix}$ in the span of $\{\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4\}$? Justify your answer.

Solution: Letting $\vec{u} = \begin{bmatrix} 6 \\ 5 \\ 7 \\ -7 \end{bmatrix}$, the answer to this question is yes if the system of linear equations

$[\vec{v}_1 \ \vec{v}_2 \ \vec{v}_3 \ \vec{v}_4 \mid \vec{u}]$ has at least one solution. To determine the number of solutions, find the reduced row echelon form:

$$\left[\begin{array}{cccc|c} 0 & -1 & -5 & 15 & 6 \\ -4 & -1 & 0 & 15 & 5 \\ 1 & -5 & -3 & -18 & 7 \\ 2 & 1 & 4 & -21 & -7 \end{array} \right] \xrightarrow{\text{RREF}} \left[\begin{array}{cccc|c} 1 & 0 & 0 & -5 & -1 \\ 0 & 1 & 0 & 5 & -1 \\ 0 & 0 & 1 & -4 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right]$$

There are infinitely many solutions to this system of linear equations, so \vec{u} is in the subspace.

Grading: +3 points for the setting up the matrix, +3 points for the RREF, +4 points for the answer and justification.

242 Test 2 B

SOLUTIONS

1. [30 points] For the matrix A below, find a basis for the null space of A , a basis for the row space of A , a basis for the column space of A , the rank of A , and the nullity of A . The reduced row echelon form of A is the matrix R given below.

$$A = \begin{bmatrix} 3 & 15 & 12 & -12 & -2 & 5 \\ -3 & -15 & -12 & 12 & 1 & -4 \\ 2 & 10 & 8 & -8 & 0 & 2 \end{bmatrix} \qquad R = \begin{bmatrix} 1 & 5 & 4 & -4 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Solution: To find a basis for the null space of A , you need to solve the equation $A\vec{x} = \vec{0}$ and parameterize the solutions. Looking at the reduced row echelon form, it is easy to see that $x_2 = r$, $x_3 = s$, $x_4 = t$, and $x_6 = u$ are free variables; then $x_1 = -5r - 4s + 4t - u$ and $x_5 = u$, so a vector in the null space is of the form

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = r \cdot \begin{bmatrix} -5 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + s \cdot \begin{bmatrix} -4 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + t \cdot \begin{bmatrix} 4 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + u \cdot \begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}$$

and $\left\{ \begin{bmatrix} -5 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -4 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 4 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} \right\}$ is a basis for the null space. The number of vectors in this

basis is 4, so the nullity of A is $\boxed{4}$.

To find a basis for the row space of A , take the nonzero rows of R :

$$\boxed{\{[1, 5, 4, -4, 0, 1], [0, 0, 0, 0, 1, -1]\}}.$$

To find a basis for the column space of A , take the columns of A which have a pivot in their column in R , namely

$$\boxed{\left\{ \begin{bmatrix} 3 \\ -3 \\ 2 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} \right\}}.$$

The number of vectors in either of these sets is the rank of A ; thus $\text{rank } A = \boxed{2}$.

Grading: +5 points for the row space basis, +5 points for the column space basis, +5 points for the nullity, +5 points for the rank, and +10 points for the null space basis. Grading for common mistakes: -2 points for choosing rows from A ; -3 points for choosing columns from R ; -3 points for the rank or nullity not equalling the number of vectors in the appropriate basis.

2. Let B be the (ordered) basis $\left(\begin{bmatrix} -1 \\ 0 \\ -3 \end{bmatrix}, \begin{bmatrix} -3 \\ -1 \\ -12 \end{bmatrix}, \begin{bmatrix} -2 \\ -3 \\ -14 \end{bmatrix} \right)$ and C the basis $\left(\begin{bmatrix} -1 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix}, \begin{bmatrix} -2 \\ 4 \\ 5 \end{bmatrix} \right)$.

a. [15 points] Find the coordinates of $\begin{bmatrix} -18 \\ -10 \\ -82 \end{bmatrix}$ with respect to the basis B .

Solution: Letting $\vec{u} = \begin{bmatrix} -18 \\ -10 \\ -82 \end{bmatrix}$,

$$[\vec{u}]_B = \tilde{B}^{-1} \cdot \vec{u} = \begin{bmatrix} -1 & -3 & -2 \\ 0 & -1 & -3 \\ -3 & -12 & -14 \end{bmatrix}^{-1} \cdot \begin{bmatrix} -18 \\ -10 \\ -82 \end{bmatrix} = \boxed{\begin{bmatrix} 2 \\ 4 \\ 2 \end{bmatrix}}.$$

Grading: +5 points for the formula, +5 points for substitution, +5 points for calculation.

b. [15 points] If the coordinates of \vec{u} with respect to C are $\begin{bmatrix} 3 \\ 13 \\ 12 \end{bmatrix}$, what are the coordinates of \vec{u} with respect to B ?

Solution:

$$[\vec{u}]_B = \tilde{B}^{-1} \cdot \tilde{C} \cdot [\vec{u}]_C = \begin{bmatrix} -1 & -3 & -2 \\ 0 & -1 & -3 \\ -3 & -12 & -14 \end{bmatrix}^{-1} \cdot \begin{bmatrix} -1 & -1 & -2 \\ 1 & 2 & 4 \\ 3 & 2 & 5 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ 13 \\ 12 \end{bmatrix} = \boxed{\begin{bmatrix} 159 \\ -29 \\ -16 \end{bmatrix}}.$$

Grading: +5 points for the formula, +5 points for substitution, +5 points for calculation.

Grading for common mistakes: +10 points (total) for $\begin{bmatrix} 181 \\ 1177 \\ -646 \end{bmatrix}$ (backwards); +7 points (total) for

$\begin{bmatrix} 193 \\ -170 \\ -638 \end{bmatrix} = CB^{-1}[u]$; +7 points (total) for $\begin{bmatrix} 119 \\ -53 \\ 235 \end{bmatrix} = BC^{-1}[u]$; +7 points (total) for $B^{-1}[u]$ or $C^{-1}[u]$.

3. Let $\vec{v}_1 = \begin{bmatrix} -5 \\ 1 \\ 0 \\ -5 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} 2 \\ 4 \\ -3 \\ -3 \end{bmatrix}$, $\vec{v}_3 = \begin{bmatrix} -10 \\ 2 \\ 0 \\ -10 \end{bmatrix}$, and $\vec{v}_4 = \begin{bmatrix} 4 \\ -3 \\ 2 \\ 5 \end{bmatrix}$.

- a. [15 points] The set $\{\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4\}$ is linearly dependent. Find a nontrivial linear combination of these vectors that adds up to $\vec{0}$.

Solution: To do this, you need to find a nontrivial solution to $c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3 + c_4\vec{v}_4 = \vec{0}$, whose matrix looks like $[\vec{v}_1 \ \vec{v}_2 \ \vec{v}_3 \ \vec{v}_4 \mid \vec{0}]$, or

$$\left[\begin{array}{cccc|c} -5 & 2 & -10 & 4 & 0 \\ 1 & 4 & 2 & -3 & 0 \\ 0 & -3 & 0 & 2 & 0 \\ -5 & -3 & -10 & 5 & 0 \end{array} \right] \xrightarrow{\text{RREF}} \left[\begin{array}{cccc|c} 1 & 0 & 2 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right].$$

The solutions to this system are parameterized by

$$\begin{aligned} c_1 &= -2\alpha \\ c_2 &= 0 \\ c_3 &= \alpha \\ c_4 &= 0 \end{aligned}$$

where α can be any real number. Choosing $\alpha = 1$ produces the linear combination

$$\boxed{-2\vec{v}_1 + \vec{v}_3 = \vec{0}}$$

(but choosing any other nonzero real value for α will also produce a correct answer).

Grading: +5 points for setting up the system of linear equations; +5 points for parameterizing it; +5 points for finding a linear combination. *Grading for common mistakes:* +7 points (total) for using the “column space” method instead.

- b. [15 points] Find a basis for the subspace spanned by $\{\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4\}$, and the dimension of that subspace.

Solution: To find a basis, choose the vectors which have a pivot in their columns in the reduced

row echelon form. This basis is $\boxed{\{\vec{v}_1, \vec{v}_2, \vec{v}_4\}}$ or $\left\{ \begin{bmatrix} -5 \\ 1 \\ 0 \\ -5 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \\ -3 \\ -3 \end{bmatrix}, \begin{bmatrix} 4 \\ -3 \\ 2 \\ 5 \end{bmatrix} \right\}$. The dimension of the

subspace is the number of vectors in the basis, namely $\boxed{3}$.

Grading: +10 points for the basis, +5 points for the dimension. *Grading for common mistakes:* +7 points (total) for using the “null space” method instead.

c. [10 points] Is the vector $\begin{bmatrix} -1 \\ 30 \\ 36 \\ 19 \end{bmatrix}$ in the span of $\{\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4\}$? Justify your answer.

Solution: Letting $\vec{u} = \begin{bmatrix} -1 \\ 30 \\ 36 \\ 19 \end{bmatrix}$, the answer to this question is yes if the system of linear equations

$[\vec{v}_1 \ \vec{v}_2 \ \vec{v}_3 \ \vec{v}_4 \mid \vec{u}]$ has at least one solution. To determine the number of solutions, find the reduced row echelon form:

$$\left[\begin{array}{cccc|c} -5 & 2 & -10 & 4 & -1 \\ 1 & 4 & 2 & -3 & 30 \\ 0 & -3 & 0 & 2 & 36 \\ -5 & -3 & -10 & 5 & 19 \end{array} \right] \xrightarrow{\text{RREF}} \left[\begin{array}{cccc|c} 1 & 0 & 2 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{array} \right]$$

There are no solutions to this system of linear equations, so \vec{u} is not in the subspace.

Grading: +3 points for the setting up the matrix, +3 points for the RREF, +4 points for the answer and justification.

242 Test 2 C

SOLUTIONS

1. [30 points] For the matrix A below, find a basis for the null space of A , a basis for the row space of A , a basis for the column space of A , the rank of A , and the nullity of A . The reduced row echelon form of A is the matrix R given below.

$$A = \begin{bmatrix} 1 & 0 & -2 & 3 & 3 \\ 2 & -1 & 1 & 10 & 3 \\ -2 & -2 & 14 & 2 & -12 \end{bmatrix} \qquad R = \begin{bmatrix} 1 & 0 & -2 & 3 & 3 \\ 0 & 1 & -5 & -4 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Solution: To find a basis for the null space of A , you need to solve the equation $A\vec{x} = \vec{0}$ and parameterize the solutions. Looking at the reduced row echelon form, it is easy to see that $x_3 = r$, $x_4 = s$, and $x_5 = t$ are free variables; then $x_1 = 2r - 3s - 3t$ and $x_2 = 5r + 4s - 3t$, so a vector in the null space is of the form

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = r \cdot \begin{bmatrix} 2 \\ 5 \\ 1 \\ 0 \\ 0 \end{bmatrix} + s \cdot \begin{bmatrix} -3 \\ 4 \\ 0 \\ 1 \\ 0 \end{bmatrix} + t \cdot \begin{bmatrix} -3 \\ -3 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

and $\left\{ \begin{bmatrix} 2 \\ 5 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 4 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ -3 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$ is a basis for the null space. The number of vectors in this basis is

3, so the nullity of A is $\boxed{3}$.

To find a basis for the row space of A , take the nonzero rows of R :

$$\boxed{\{[1, 0, -2, 3, 3], [0, 1, -5, -4, 3]\}}.$$

To find a basis for the column space of A , take the columns of A which have a pivot in their column in R , namely

$$\boxed{\left\{ \begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ -2 \end{bmatrix} \right\}}.$$

The number of vectors in either of these sets is the rank of A ; thus $\text{rank } A = \boxed{2}$.

Grading: +5 points for the row space basis, +5 points for the column space basis, +5 points for the nullity, +5 points for the rank, and +10 points for the null space basis. Grading for common mistakes: -2 points for choosing rows from A ; -3 points for choosing columns from R ; -3 points for the rank or nullity not equalling the number of vectors in the appropriate basis.

2. Let B be the (ordered) basis $\left(\begin{bmatrix} -1 \\ -2 \\ 3 \end{bmatrix}, \begin{bmatrix} -2 \\ -3 \\ 6 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right)$ and C the basis $\left(\begin{bmatrix} 1 \\ -3 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -2 \end{bmatrix} \right)$.

- a. [10 points] Find the coordinates of $\begin{bmatrix} 6 \\ -12 \\ 10 \end{bmatrix}$ with respect to the basis C .

Solution: Letting $\vec{u} = \begin{bmatrix} 6 \\ -12 \\ 10 \end{bmatrix}$,

$$[\vec{u}]_C = \tilde{C}^{-1} \cdot \vec{u} = \begin{bmatrix} 1 & 1 & 0 \\ -3 & -2 & 1 \\ 3 & 2 & -2 \end{bmatrix}^{-1} \cdot \begin{bmatrix} 6 \\ -12 \\ 10 \end{bmatrix} = \boxed{\begin{bmatrix} 2 \\ 4 \\ 2 \end{bmatrix}}.$$

Grading: +5 points for the formula, +5 points for substitution, +5 points for calculation.

- b. [15 points] If the coordinates of \vec{u} with respect to B are $\begin{bmatrix} 3 \\ 13 \\ 12 \end{bmatrix}$, what are the coordinates of \vec{u} with respect to C ?

Solution:

$$[\vec{u}]_C = \tilde{C}^{-1} \cdot \tilde{B} \cdot [\vec{u}]_B = \begin{bmatrix} 1 & 1 & 0 \\ -3 & -2 & 1 \\ 3 & 2 & -2 \end{bmatrix}^{-1} \cdot \begin{bmatrix} -1 & -2 & 0 \\ -2 & -3 & 0 \\ 3 & 6 & 1 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ 13 \\ 12 \end{bmatrix} = \boxed{\begin{bmatrix} 49 \\ -78 \\ -54 \end{bmatrix}}.$$

Grading: +5 points for the formula, +5 points for substitution, +5 points for calculation.

Grading for common mistakes: +10 points (total) for $\begin{bmatrix} 94 \\ -55 \\ 59 \end{bmatrix}$ (backwards); +7 points (total) for

$\begin{bmatrix} -10 \\ 58 \\ -79 \end{bmatrix} = CB^{-1}[u]$; +7 points (total) for $\begin{bmatrix} -50 \\ -53 \\ 125 \end{bmatrix} = BC^{-1}[u]$; +7 points (total) for $B^{-1}[u]$ or $C^{-1}[u]$.

3. Let $\vec{v}_1 = \begin{bmatrix} -2 \\ 2 \\ -3 \\ 1 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} 1 \\ 5 \\ -3 \\ -1 \end{bmatrix}$, $\vec{v}_3 = \begin{bmatrix} 5 \\ 13 \\ -6 \\ -4 \end{bmatrix}$, and $\vec{v}_4 = \begin{bmatrix} -3 \\ -5 \\ -5 \\ -1 \end{bmatrix}$.

- a. [15 points] The set $\{\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4\}$ is linearly dependent. Find a nontrivial linear combination of these vectors that adds up to $\vec{0}$.

Solution: To do this, you need to find a nontrivial solution to $c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3 + c_4\vec{v}_4 = \vec{0}$, whose matrix looks like $[\vec{v}_1 \ \vec{v}_2 \ \vec{v}_3 \ \vec{v}_4 \mid \vec{0}]$, or

$$\left[\begin{array}{cccc|c} -2 & 1 & 5 & -3 & 0 \\ 2 & 5 & 13 & -5 & 0 \\ -3 & -3 & -6 & -5 & 0 \\ 1 & -1 & -4 & -1 & 0 \end{array} \right] \xrightarrow{\text{RREF}} \left[\begin{array}{cccc|c} 1 & 0 & -1 & 0 & 0 \\ 0 & 1 & 3 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right].$$

The solutions to this system are parameterized by

$$\begin{aligned} c_1 &= \alpha \\ c_2 &= -3\alpha \\ c_3 &= \alpha \\ c_4 &= 0 \end{aligned}$$

where α can be any real number. Choosing $\alpha = 1$ produces the linear combination

$$\boxed{\vec{v}_1 - 3\vec{v}_2 + \vec{v}_3 = \vec{0}}$$

(but choosing any other nonzero real value for α will also produce a correct answer).

Grading: +5 points for setting up the system of linear equations; +5 points for parameterizing it; +5 points for finding a linear combination. Grading for common mistakes: +7 points (total) for using the “column space” method instead.

- b. [15 points] Find a basis for the subspace spanned by $\{\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4\}$, and the dimension of that subspace.

Solution: To find a basis, choose the vectors which have a pivot in their columns in the reduced

row echelon form. This basis is $\boxed{\{\vec{v}_1, \vec{v}_2, \vec{v}_4\}}$ or $\left\{ \begin{bmatrix} -2 \\ 2 \\ -3 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 5 \\ -3 \\ -1 \end{bmatrix}, \begin{bmatrix} -3 \\ -5 \\ -5 \\ -1 \end{bmatrix} \right\}$. The dimension of the

subspace is the number of vectors in the basis, namely $\boxed{3}$.

Grading: +10 points for the basis, +5 points for the dimension. Grading for common mistakes: +7 points (total) for using the “null space” method instead.

c. [10 points] Is the vector $\begin{bmatrix} 41 \\ -5 \\ -27 \\ 33 \end{bmatrix}$ in the span of $\{\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4\}$? Justify your answer.

Solution: Letting $\vec{u} = \begin{bmatrix} 41 \\ -5 \\ -27 \\ 33 \end{bmatrix}$, the answer to this question is yes if the system of linear equations

$[\vec{v}_1 \ \vec{v}_2 \ \vec{v}_3 \ \vec{v}_4 \mid \vec{u}]$ has at least one solution. To determine the number of solutions, find the reduced row echelon form:

$$\left[\begin{array}{cccc|c} -2 & 1 & 5 & -3 & 41 \\ 2 & 5 & 13 & -5 & -5 \\ -3 & -3 & -6 & -5 & -27 \\ 1 & -1 & -4 & -1 & 33 \end{array} \right] \xrightarrow{\text{RREF}} \left[\begin{array}{cccc|c} 1 & 0 & -1 & 0 & 0 \\ 0 & 1 & 3 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{array} \right]$$

There are no solutions to this system of linear equations, so \vec{u} is not in the subspace.

Grading: +3 points for the setting up the matrix, +3 points for the RREF, +4 points for the answer and justification.

1. [30 points] For the matrix A below, find a basis for the null space of A , a basis for the row space of A , a basis for the column space of A , the rank of A , and the nullity of A . The reduced row echelon form of A is the matrix R given below.

$$A = \begin{bmatrix} -2 & -10 & 10 & -2 & 12 \\ -4 & -20 & 20 & 2 & -6 \\ -1 & -5 & 5 & -4 & 21 \end{bmatrix} \qquad R = \begin{bmatrix} 1 & 5 & -5 & 0 & -1 \\ 0 & 0 & 0 & 1 & -5 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Solution: To find a basis for the null space of A , you need to solve the equation $A\vec{x} = \vec{0}$ and parameterize the solutions. Looking at the reduced row echelon form, it is easy to see that $x_2 = r$, $x_3 = s$, and $x_5 = t$ are free variables; then $x_1 = -5r + 5s + t$ and $x_4 = 5t$, so a vector in the null space is of the form

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = r \cdot \begin{bmatrix} -5 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + s \cdot \begin{bmatrix} 5 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t \cdot \begin{bmatrix} 1 \\ 0 \\ 0 \\ 5 \\ 1 \end{bmatrix}$$

and $\left\{ \begin{bmatrix} -5 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 5 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 5 \\ 1 \end{bmatrix} \right\}$ is a basis for the null space. The number of vectors in this basis is

3, so the nullity of A is $\boxed{3}$.

To find a basis for the row space of A , take the nonzero rows of R :

$$\{[1, 5, -5, 0, -1], [0, 0, 0, 1, -5]\}.$$

To find a basis for the column space of A , take the columns of A which have a pivot in their column in R , namely

$$\left\{ \begin{bmatrix} -2 \\ -4 \\ -1 \end{bmatrix}, \begin{bmatrix} -2 \\ 2 \\ -4 \end{bmatrix} \right\}.$$

The number of vectors in either of these sets is the rank of A ; thus $\text{rank } A = \boxed{2}$.

Grading: +5 points for the row space basis, +5 points for the column space basis, +5 points for the nullity, +5 points for the rank, and +10 points for the null space basis. Grading for common mistakes: -2 points for choosing rows from A ; -3 points for choosing columns from R ; -3 points for the rank or nullity not equalling the number of vectors in the appropriate basis.

2. Let B be the (ordered) basis $\left(\begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ -4 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ -5 \end{bmatrix}\right)$ and C the basis $\left(\begin{bmatrix} 1 \\ -3 \\ -1 \end{bmatrix}, \begin{bmatrix} -2 \\ 7 \\ 5 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ -4 \end{bmatrix}\right)$.

a. [10 points] Find the coordinates of $\begin{bmatrix} -23 \\ 70 \\ 14 \end{bmatrix}$ with respect to the basis C .

Solution: Letting $\vec{u} = \begin{bmatrix} -23 \\ 70 \\ 14 \end{bmatrix}$,

$$[\vec{u}]_C = \tilde{C}^{-1} \cdot \vec{u} = \begin{bmatrix} 1 & -2 & 0 \\ -3 & 7 & -1 \\ -1 & 5 & -4 \end{bmatrix}^{-1} \cdot \begin{bmatrix} -23 \\ 70 \\ 14 \end{bmatrix} = \boxed{\begin{bmatrix} 3 \\ 13 \\ 12 \end{bmatrix}}.$$

Grading: +5 points for the formula, +5 points for substitution, +5 points for calculation.

b. [15 points] If the coordinates of \vec{u} with respect to C are $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, what are the coordinates of \vec{u} with respect to B ?

Solution:

$$[\vec{u}]_B = \tilde{B}^{-1} \cdot \tilde{C} \cdot [\vec{u}]_C = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ -2 & -4 & -5 \end{bmatrix}^{-1} \cdot \begin{bmatrix} 1 & -2 & 0 \\ -3 & 7 & -1 \\ -1 & 5 & -4 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \boxed{\begin{bmatrix} -25 \\ 22 \\ -7 \end{bmatrix}}.$$

Grading: +5 points for the formula, +5 points for substitution, +5 points for calculation.

Grading for common mistakes: +10 points (total) for $\begin{bmatrix} 183 \\ 90 \\ 73 \end{bmatrix}$ (backwards); +7 points (total) for

$\begin{bmatrix} -59 \\ 206 \\ 155 \end{bmatrix} = CB^{-1}[u]$; +7 points (total) for $\begin{bmatrix} 49 \\ 38 \\ -185 \end{bmatrix} = BC^{-1}[u]$; +7 points (total) for $B^{-1}[u]$ or $C^{-1}[u]$.

3. Let $\vec{v}_1 = \begin{bmatrix} 3 \\ -4 \\ -4 \\ -1 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} -15 \\ 20 \\ 20 \\ 5 \end{bmatrix}$, $\vec{v}_3 = \begin{bmatrix} -3 \\ -5 \\ -4 \\ -3 \end{bmatrix}$, and $\vec{v}_4 = \begin{bmatrix} -5 \\ 5 \\ 3 \\ -3 \end{bmatrix}$.

- a. [15 points] The set $\{\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4\}$ is linearly dependent. Find a nontrivial linear combination of these vectors that adds up to $\vec{0}$.

Solution: To do this, you need to find a nontrivial solution to $c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3 + c_4\vec{v}_4 = \vec{0}$, whose matrix looks like $[\vec{v}_1 \ \vec{v}_2 \ \vec{v}_3 \ \vec{v}_4 \mid \vec{0}]$, or

$$\left[\begin{array}{cccc|c} 3 & -15 & -3 & -5 & 0 \\ -4 & 20 & -5 & 5 & 0 \\ -4 & 20 & -4 & 3 & 0 \\ -1 & 5 & -3 & -3 & 0 \end{array} \right] \xrightarrow{\text{RREF}} \left[\begin{array}{cccc|c} 1 & -5 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right].$$

The solutions to this system are parameterized by

$$\begin{aligned} c_1 &= 5\alpha \\ c_2 &= \alpha \\ c_3 &= 0 \\ c_4 &= 0 \end{aligned}$$

where α can be any real number. Choosing $\alpha = 1$ produces the linear combination

$$\boxed{5\vec{v}_1 + \vec{v}_2 = \vec{0}}$$

(but choosing any other nonzero real value for α will also produce a correct answer).

Grading: +5 points for setting up the system of linear equations; +5 points for parameterizing it; +5 points for finding a linear combination. *Grading for common mistakes:* +7 points (total) for using the “column space” method instead.

- b. [15 points] Find a basis for the subspace spanned by $\{\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4\}$, and the dimension of that subspace.

Solution: To find a basis, choose the vectors which have a pivot in their columns in the reduced

row echelon form. This basis is $\boxed{\{\vec{v}_1, \vec{v}_3, \vec{v}_4\}}$ or $\left\{ \begin{bmatrix} 3 \\ -4 \\ -4 \\ -1 \end{bmatrix}, \begin{bmatrix} -3 \\ -5 \\ -4 \\ -3 \end{bmatrix}, \begin{bmatrix} -5 \\ 5 \\ 3 \\ -3 \end{bmatrix} \right\}$. The dimension of the

subspace is the number of vectors in the basis, namely $\boxed{3}$.

Grading: +10 points for the basis, +5 points for the dimension. *Grading for common mistakes:* +7 points (total) for using the “null space” method instead.

c. [10 points] Is the vector $\begin{bmatrix} 48 \\ -98 \\ 79 \\ -64 \end{bmatrix}$ in the span of $\{\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4\}$? Justify your answer.

Solution: Letting $\vec{u} = \begin{bmatrix} 48 \\ -98 \\ 79 \\ -64 \end{bmatrix}$, the answer to this question is yes if the system of linear equations

$[\vec{v}_1 \ \vec{v}_2 \ \vec{v}_3 \ \vec{v}_4 \mid \vec{u}]$ has at least one solution. To determine the number of solutions, find the reduced row echelon form:

$$\left[\begin{array}{cccc|c} 3 & -15 & -3 & -5 & 48 \\ -4 & 20 & -5 & 5 & -98 \\ -4 & 20 & -4 & 3 & 79 \\ -1 & 5 & -3 & -3 & -64 \end{array} \right] \xrightarrow{\text{RREF}} \left[\begin{array}{cccc|c} 1 & -5 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{array} \right]$$

There are no solutions to this system of linear equations, so \vec{u} is not in the subspace.

Grading: +3 points for the setting up the matrix, +3 points for the RREF, +4 points for the answer and justification.

242 Test 2

MAKE-UP SOLUTIONS

1. [30 points] For the matrix A below, find a basis for the null space of A , a basis for the row space of A , a basis for the column space of A , the rank of A , and the nullity of A . The reduced row echelon form of A is the matrix R given below.

$$A = \begin{bmatrix} 1 & 4 & -3 & -1 \\ 0 & 0 & 0 & 0 \\ 2 & 8 & -6 & -2 \end{bmatrix} \qquad R = \begin{bmatrix} 1 & 4 & -3 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Solution: To find a basis for the null space of A , you need to solve the equation $A\vec{x} = \vec{0}$ and parameterize the solutions. Looking at the reduced row echelon form, it is easy to see that $x_2 = r$, $x_3 = s$, and $x_4 = t$ are free variables; then $x_1 = -4r + 3s + t$, so a vector in the null space is of the form

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = r \cdot \begin{bmatrix} -4 \\ 1 \\ 0 \\ 0 \end{bmatrix} + s \cdot \begin{bmatrix} 3 \\ 0 \\ 1 \\ 0 \end{bmatrix} + t \cdot \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

and $\left\{ \begin{bmatrix} -4 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$ is a basis for the null space. The number of vectors in this basis is

3, so the nullity of A is $\boxed{3}$.

To find a basis for the row space of A , take the nonzero rows of R :

$$\boxed{\{[1, 4, -3, -1]\}}.$$

To find a basis for the column space of A , take the columns of A which have a pivot in their column in R , namely

$$\boxed{\left\{ \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} \right\}}.$$

The number of vectors in either of these sets is the rank of A ; thus $\text{rank } A = \boxed{1}$.

Grading: +5 points for the row space basis, +5 points for the column space basis, +5 points for the nullity, +5 points for the rank, and +10 points for the null space basis. Grading for common mistakes: -2 points for choosing rows from A ; -3 points for choosing columns from R ; -3 points for the rank or nullity not equalling the number of vectors in the appropriate basis.

2. Let B be the (ordered) basis $\left(\left[\begin{array}{c} -1 \\ 3 \\ 2 \end{array}\right], \left[\begin{array}{c} 2 \\ -7 \\ -7 \end{array}\right], \left[\begin{array}{c} 0 \\ -1 \\ -2 \end{array}\right]\right)$ and C the basis $\left(\left[\begin{array}{c} -1 \\ 3 \\ 2 \end{array}\right], \left[\begin{array}{c} -1 \\ 4 \\ 4 \end{array}\right], \left[\begin{array}{c} 0 \\ 0 \\ -1 \end{array}\right]\right)$.

a. [15 (NOT 10) points] Find the coordinates of $\begin{bmatrix} 6 \\ -24 \\ -28 \end{bmatrix}$ with respect to the basis B .

Solution: Letting $\vec{u} = \begin{bmatrix} 6 \\ -24 \\ -28 \end{bmatrix}$,

$$[\vec{u}]_B = \tilde{B}^{-1} \cdot \vec{u} = \begin{bmatrix} -1 & 2 & 0 \\ 3 & -7 & -1 \\ 2 & -7 & -2 \end{bmatrix}^{-1} \cdot \begin{bmatrix} 6 \\ -24 \\ -28 \end{bmatrix} = \boxed{\begin{bmatrix} 2 \\ 4 \\ 2 \end{bmatrix}}.$$

Grading: +5 points for the formula, +5 points for substitution, +5 points for calculation.

b. [15 points] If the coordinates of \vec{u} with respect to B are $\begin{bmatrix} 3 \\ 13 \\ 12 \end{bmatrix}$, what are the coordinates of \vec{u} with respect to C ?

Solution:

$$[\vec{u}]_C = \tilde{C}^{-1} \cdot \tilde{B} \cdot [\vec{u}]_B = \begin{bmatrix} -1 & -1 & 0 \\ 3 & 4 & 0 \\ 2 & 4 & -1 \end{bmatrix}^{-1} \cdot \begin{bmatrix} -1 & 2 & 0 \\ 3 & -7 & -1 \\ 2 & -7 & -2 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ 13 \\ 12 \end{bmatrix} = \boxed{\begin{bmatrix} 2 \\ -25 \\ 13 \end{bmatrix}}.$$

Grading: +5 points for the formula, +5 points for substitution, +5 points for calculation.

Grading for common mistakes: +10 points (total) for $\begin{bmatrix} 40 \\ 12 \\ -25 \end{bmatrix}$ (backwards); +7 points (total) for

$\begin{bmatrix} -75 \\ 251 \\ 250 \end{bmatrix} = CB^{-1}[u]$; +7 points (total) for $\begin{bmatrix} 69 \\ -255 \\ -256 \end{bmatrix} = BC^{-1}[u]$; +7 points (total) for $B^{-1}[u]$ or $C^{-1}[u]$.

3. Let $\vec{v}_1 = \begin{bmatrix} 0 \\ -2 \\ -1 \\ 3 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} 4 \\ -1 \\ 3 \\ 4 \end{bmatrix}$, $\vec{v}_3 = \begin{bmatrix} 8 \\ 4 \\ 9 \\ -1 \end{bmatrix}$, and $\vec{v}_4 = \begin{bmatrix} -4 \\ 1 \\ -1 \\ -5 \end{bmatrix}$.

- a. [15 points] The set $\{\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4\}$ is linearly dependent. Find a nontrivial linear combination of these vectors that adds up to $\vec{0}$.

Solution: To do this, you need to find a nontrivial solution to $c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3 + c_4\vec{v}_4 = \vec{0}$, whose matrix looks like $[\vec{v}_1 \ \vec{v}_2 \ \vec{v}_3 \ \vec{v}_4 \mid \vec{0}]$, or

$$\left[\begin{array}{cccc|c} 0 & 4 & 8 & -4 & 0 \\ -2 & -1 & 4 & 1 & 0 \\ -1 & 3 & 9 & -1 & 0 \\ 3 & 4 & -1 & -5 & 0 \end{array} \right] \xrightarrow{\text{RREF}} \left[\begin{array}{cccc|c} 1 & 0 & -3 & 0 & 0 \\ 0 & 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right].$$

The solutions to this system are parameterized by

$$\begin{aligned} c_1 &= 3\alpha \\ c_2 &= -2\alpha \\ c_3 &= \alpha \\ c_4 &= 0 \end{aligned}$$

where α can be any real number. Choosing $\alpha = 1$ produces the linear combination

$$\boxed{3\vec{v}_1 - 2\vec{v}_2 + \vec{v}_3 = \vec{0}}$$

(but choosing any other nonzero real value for α will also produce a correct answer).

Grading: +5 points for setting up the system of linear equations; +5 points for parameterizing it; +5 points for finding a linear combination. Grading for common mistakes: +7 points (total) for using the “column space” method instead.

- b. [15 points] Find a basis for the subspace spanned by $\{\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4\}$, and the dimension of that subspace.

Solution: To find a basis, choose the vectors which have a pivot in their columns in the reduced

row echelon form. This basis is $\boxed{\{\vec{v}_1, \vec{v}_2, \vec{v}_4\}}$ or $\left\{ \begin{bmatrix} 0 \\ -2 \\ -1 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ -1 \\ 3 \\ 4 \end{bmatrix}, \begin{bmatrix} -4 \\ 1 \\ -1 \\ -5 \end{bmatrix} \right\}$. The dimension of the

subspace is the number of vectors in the basis, namely $\boxed{3}$.

Grading: +10 points for the basis, +5 points for the dimension. Grading for common mistakes: +7 points (total) for using the “null space” method instead.

c. [10 points] Is the vector $\begin{bmatrix} 17 \\ -22 \\ -11 \\ -12 \end{bmatrix}$ in the span of $\{\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4\}$? Justify your answer.

Solution: Letting $\vec{u} = \begin{bmatrix} 17 \\ -22 \\ -11 \\ -12 \end{bmatrix}$, the answer to this question is yes if the system of linear equations

$[\vec{v}_1 \ \vec{v}_2 \ \vec{v}_3 \ \vec{v}_4 \mid \vec{u}]$ has at least one solution. To determine the number of solutions, find the reduced row echelon form:

$$\left[\begin{array}{cccc|c} 0 & 4 & 8 & -4 & 17 \\ -2 & -1 & 4 & 1 & -22 \\ -1 & 3 & 9 & -1 & -11 \\ 3 & 4 & -1 & -5 & -12 \end{array} \right] \xrightarrow{\text{RREF}} \left[\begin{array}{cccc|c} 1 & 0 & -3 & 0 & 0 \\ 0 & 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{array} \right]$$

There are no solutions to this system of linear equations, so \vec{u} is not in the subspace.

Grading: +3 points for the setting up the matrix, +3 points for the RREF, +4 points for the answer and justification.