

242 Test 1 A

SOLUTIONS

1. [15 points] How many solutions does each of the following systems of linear equations have? Circle the entries which led you to your conclusion.

$$(a) \left[\begin{array}{cccc|c} \mathbf{1} & 0 & 0 & 0 & 4 \\ 0 & \mathbf{1} & 0 & 0 & 2 \\ 0 & 0 & \mathbf{1} & 0 & -3 \\ 0 & 0 & 0 & \mathbf{1} & -5 \\ 0 & 0 & 0 & 0 & \mathbf{1} & 0 \end{array} \right]$$

Solution: 1

$$(b) \left[\begin{array}{cccc|c} 1 & 0 & 0 & \mathbf{-1} & 0 \\ 0 & 1 & 0 & \mathbf{0} & 4 \\ 0 & 0 & 1 & \mathbf{1} & 2 \\ 0 & 0 & 0 & \mathbf{0} & 0 \end{array} \right]$$

Solution: ∞

$$(c) \left[\begin{array}{ccc|c} 1 & 0 & 5 & 4 \\ 0 & 1 & -1 & -3 \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{5} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

Solution: 0

Solution: Correct answers are shown above, with the relevant entries in boldface.

Grading: +3 points for the correct answer, +2 points for the work; for each part. Grading for common mistakes: -2 points for “many” (instead of “infinitely many”).

2. [15 points] Solve the system of linear equations

$$\begin{aligned} 161x_1 - 64x_2 + 6x_3 - 19x_4 &= 290 \\ -121x_1 + 48x_2 - 5x_3 + 14x_4 &= -219 \\ -25x_1 + 10x_2 - x_3 + 3x_4 &= -45 \\ 8x_1 - 3x_2 - x_4 &= 14 \end{aligned}$$

using the fact that $\begin{bmatrix} 161 & -64 & 6 & -19 \\ -121 & 48 & -5 & 14 \\ -25 & 10 & -1 & 3 \\ 8 & -3 & 0 & -1 \end{bmatrix}^{-1} = \begin{bmatrix} -1 & -1 & -1 & 2 \\ -4 & -3 & -9 & 7 \\ -3 & -2 & -9 & 2 \\ 4 & 1 & 19 & -6 \end{bmatrix}$. (Other methods may result in the loss of points.)

Solution:

$$X = A^{-1} \cdot B = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 161 & -64 & 6 & -19 \\ -121 & 48 & -5 & 14 \\ -25 & 10 & -1 & 3 \\ 8 & -3 & 0 & -1 \end{bmatrix}^{-1} \cdot \begin{bmatrix} 290 \\ -219 \\ -45 \\ 14 \end{bmatrix} = \begin{bmatrix} -1 & -1 & -1 & 2 \\ -4 & -3 & -9 & 7 \\ -3 & -2 & -9 & 2 \\ 4 & 1 & 19 & -6 \end{bmatrix} \cdot \begin{bmatrix} 290 \\ -219 \\ -45 \\ 14 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 1 \\ 2 \end{bmatrix}.$$

Grading for common mistakes: +3 points (total) for another method and a wrong answer; +5 points (total) for another method and a correct answer; -5 points for bad matrix multiplication.

3. [20 points] Find $\begin{vmatrix} 0 & 4 & 4 & 0 \\ 3 & -1 & 3 & -3 \\ 4 & 3 & 2 & -4 \\ -1 & 0 & 0 & 0 \end{vmatrix}$

Solution: Typical derivations include the following. Using Expansion by Minors:

$$\begin{vmatrix} 0 & 4 & 4 & 0 \\ 3 & -1 & 3 & -3 \\ 4 & 3 & 2 & -4 \\ -1 & 0 & 0 & 0 \end{vmatrix} = -(-1) \cdot \begin{vmatrix} 4 & 4 & 0 \\ -1 & 3 & -3 \\ 3 & 2 & -4 \end{vmatrix} = [(-48) + (-36) + 0] - [0 + (-24) + (16)] = \boxed{-76.}$$

(Note that you can use Sarrus's Method right after Expansion by Minors.)

Using Gaussian Elimination:

$$\begin{aligned} & \begin{vmatrix} 0 & 4 & 4 & 0 \\ 3 & -1 & 3 & -3 \\ 4 & 3 & 2 & -4 \\ -1 & 0 & 0 & 0 \end{vmatrix} \xrightarrow{\textcircled{1} - \textcircled{4}} \begin{vmatrix} 1 & 4 & 4 & 0 \\ 3 & -1 & 3 & -3 \\ 4 & 3 & 2 & -4 \\ -1 & 0 & 0 & 0 \end{vmatrix} \xrightarrow{\begin{matrix} \textcircled{2} - 3\textcircled{1} \\ \textcircled{3} - 4\textcircled{1} \\ \textcircled{4} + \textcircled{1} \end{matrix}} \begin{vmatrix} 1 & 4 & 4 & 0 \\ 0 & -13 & -9 & -3 \\ 0 & -13 & -14 & -4 \\ 0 & 4 & 4 & 0 \end{vmatrix} \\ & \xrightarrow{\textcircled{2} + 3\textcircled{4}} \begin{vmatrix} 1 & 4 & 4 & 0 \\ 0 & -1 & 3 & -3 \\ 0 & -13 & -14 & -4 \\ 0 & 4 & 4 & 0 \end{vmatrix} \xrightarrow{\begin{matrix} \textcircled{3} - 13\textcircled{2} \\ \textcircled{4} + 4\textcircled{2} \end{matrix}} \begin{vmatrix} 1 & 4 & 4 & 0 \\ 0 & -1 & 3 & -3 \\ 0 & 0 & -53 & 35 \\ 0 & 0 & 16 & -12 \end{vmatrix} \\ & \xrightarrow{\textcircled{3} + 4\textcircled{4}} \begin{vmatrix} 1 & 4 & 4 & 0 \\ 0 & -1 & 3 & -3 \\ 0 & 0 & 11 & -13 \\ 0 & 0 & 16 & -12 \end{vmatrix} \xrightarrow{\textcircled{4} - \textcircled{3}} \begin{vmatrix} 1 & 4 & 4 & 0 \\ 0 & -1 & 3 & -3 \\ 0 & 0 & 11 & -13 \\ 0 & 0 & 5 & 1 \end{vmatrix} \\ & \xrightarrow{\textcircled{3} - 2\textcircled{4}} \begin{vmatrix} 1 & 4 & 4 & 0 \\ 0 & -1 & 3 & -3 \\ 0 & 0 & 1 & -15 \\ 0 & 0 & 5 & 1 \end{vmatrix} \xrightarrow{\textcircled{4} - 5\textcircled{3}} \begin{vmatrix} 1 & 4 & 4 & 0 \\ 0 & -1 & 3 & -3 \\ 0 & 0 & 1 & -15 \\ 0 & 0 & 0 & 76 \end{vmatrix} \\ & = (1)(-1)(1)(76) = \boxed{-76.} \end{aligned}$$

Grading for common mistakes: +5 points for using Sarrus's Method right away (on the 4×4 matrix); -5 points for not including the effects of the row operations on the determinant.

4. [20 points] Find the inverse of the matrix $\begin{bmatrix} -5 & 0 & -1 \\ -3 & -1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$

Solution: Use Gauss-Jordan Elimination:

$$\begin{aligned} \left[\begin{array}{ccc|ccc} -5 & 0 & -1 & 1 & 0 & 0 \\ -3 & -1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 \end{array} \right] &\xrightarrow{\textcircled{1} \leftrightarrow \textcircled{3}} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 0 & 1 \\ -3 & -1 & 0 & 0 & 1 & 0 \\ -5 & 0 & -1 & 1 & 0 & 0 \end{array} \right] \\ &\xrightarrow{\substack{\textcircled{2} + 3\textcircled{1} \\ \textcircled{3} + 5\textcircled{1}}} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & -1 & 0 & 0 & 1 & 3 \\ 0 & 0 & -1 & 1 & 0 & 5 \end{array} \right] \\ &\xrightarrow{\substack{-\textcircled{2} \\ -\textcircled{3}}} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & -1 & -3 \\ 0 & 0 & 1 & -1 & 0 & -5 \end{array} \right] \end{aligned}$$

Thus, the inverse is $\boxed{\begin{bmatrix} 0 & 0 & 1 \\ 0 & -1 & -3 \\ -1 & 0 & -5 \end{bmatrix}}$.

Grading: +5 points for the setup, +12 points for the row operations, +3 points for extracting the inverse.

5. [15 points] Use Cramer's Rule to solve the following system of linear equations for (x, y) .

$$\begin{aligned}x + y &= -5 \\ -2x + 4y &= -5\end{aligned}$$

Solution: Using the notation in class, $A = \begin{bmatrix} 1 & 1 \\ -2 & 4 \end{bmatrix}$, $A_1[B] = \begin{bmatrix} -5 & 1 \\ -5 & 4 \end{bmatrix}$, $A_2[B] = \begin{bmatrix} 1 & -5 \\ -2 & -5 \end{bmatrix}$, and

$$x = \frac{\det A_1[B]}{\det A} = \boxed{\frac{-15}{6}} \quad \text{and} \quad y = \frac{\det A_2[B]}{\det A} = \boxed{\frac{-15}{6}}.$$

Grading for common mistakes: +3 points (total) for another method and a wrong answer; +5 points (total) for another method and a correct answer; full credit was given if Cramer's Rule was used to find the value of one variable, and back substitution for the other.

6. [15 points] Parameterize the solutions to the system of linear equations whose matrix, in reduced row echelon form, is below. Assume that the original variables were x_1, x_2, x_3, x_4, x_5 , and x_6 .

$$\left[\begin{array}{cccccc|c} 1 & 0 & 5 & 0 & -1 & 0 & -1 \\ 0 & 1 & -3 & 0 & 4 & 0 & -19 \\ 0 & 0 & 0 & 1 & -3 & 0 & 12 \\ 0 & 0 & 0 & 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array} \right]$$

Solution: The variables x_3 and x_5 are the free variables, so let $x_3 = r$ and $x_5 = s$. Then convert the rows of the matrix into equations, solve for the lead variables, and substitute r and s to get:

$$\begin{aligned}x_1 &= -1 - 5x_3 + x_5 = -1 - 5r + s \\ x_2 &= -19 + 3x_3 - 4x_5 = -19 + 3r - 4s \\ x_4 &= 12 + 3x_5 = 12 + 3s \\ x_6 &= -2\end{aligned}$$

then collect the equations to get the answer

$$\boxed{\begin{aligned}x_1 &= -1 - 5r + s \\ x_2 &= -19 + 3r - 4s \\ x_3 &= r \\ x_4 &= 12 + 3s \\ x_5 &= s \\ x_6 &= -2 \\ \text{where } r, s &\text{ can be any real numbers}\end{aligned}}$$

Grading for common mistakes: -3 points for forgetting "where ... can be any real numbers"; -3 points for missing the equations for the free variables; -2 points for the wrong number of free variables.

242 Test 1 B

SOLUTIONS

1. [15 points] How many solutions does each of the following systems of linear equations have? Circle the entries which led you to your conclusion.

$$(a) \left[\begin{array}{ccc|c} 1 & \mathbf{4} & 0 & 15 \\ 0 & \mathbf{0} & 1 & 0 \\ 0 & \mathbf{0} & 0 & 0 \\ 0 & \mathbf{0} & 0 & 0 \\ 0 & \mathbf{0} & 0 & 0 \end{array} \right]$$

Solution: ∞

$$(b) \left[\begin{array}{ccc|c} 1 & 0 & -3 & -4 \\ 0 & 1 & -3 & 0 \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{-2} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

Solution: 0

$$(c) \left[\begin{array}{ccc|c} \mathbf{1} & 0 & 0 & 2 \\ 0 & \mathbf{1} & 0 & 4 \\ 0 & 0 & \mathbf{1} & 1 \end{array} \right]$$

Solution: 1

Solution: Correct answers are shown above, with the relevant entries in boldface.

Grading: +3 points for the correct answer, +2 points for the work; for each part. *Grading for common mistakes:* -2 points for “many” (instead of “infinitely many”).

2. [15 points] Solve the system of linear equations

$$\begin{aligned} -12x_1 - 45x_2 + 124x_3 + 27x_4 &= 460 \\ 5x_1 + 20x_2 - 55x_3 - 12x_4 &= -204 \\ -3x_2 + 9x_3 + 2x_4 &= 34 \\ -x_1 - 2x_2 + 5x_3 + x_4 &= 18 \end{aligned}$$

using the fact that $\begin{bmatrix} -12 & -45 & 124 & 27 \\ 5 & 20 & -55 & -12 \\ 0 & -3 & 9 & 2 \\ -1 & -2 & 5 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} -1 & -2 & 1 & 1 \\ 3 & 7 & 2 & -1 \\ 1 & 3 & 3 & 3 \\ 0 & -3 & -10 & -15 \end{bmatrix}$. (Other methods may result in the loss of points.)

Solution:

$$X = A^{-1} \cdot B = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -12 & -45 & 124 & 27 \\ 5 & 20 & -55 & -12 \\ 0 & -3 & 9 & 2 \\ -1 & -2 & 5 & 1 \end{bmatrix}^{-1} \cdot \begin{bmatrix} 460 \\ -204 \\ 34 \\ 18 \end{bmatrix} = \begin{bmatrix} -1 & -2 & 1 & 1 \\ 3 & 7 & 2 & -1 \\ 1 & 3 & 3 & 3 \\ 0 & -3 & -10 & -15 \end{bmatrix} \cdot \begin{bmatrix} 460 \\ -204 \\ 34 \\ 18 \end{bmatrix} = \boxed{\begin{bmatrix} 0 \\ 2 \\ 4 \\ 2 \end{bmatrix}}$$

Grading for common mistakes: +3 points (total) for another method and a wrong answer; +5 points (total) for another method and a correct answer; -5 points for bad matrix multiplication.

3. [20 points] Find $\begin{vmatrix} -2 & 5 & -4 & -4 \\ 0 & 0 & -4 & 0 \\ 0 & 4 & 2 & 0 \\ 0 & -1 & -5 & 1 \end{vmatrix}$

Solution: Typical derivations include the following. Using Expansion by Minors:

$$\begin{vmatrix} -2 & 5 & -4 & -4 \\ 0 & 0 & -4 & 0 \\ 0 & 4 & 2 & 0 \\ 0 & -1 & -5 & 1 \end{vmatrix} \xrightarrow{EM: C1} -2 \cdot \begin{vmatrix} 0 & -4 & 0 \\ 4 & 2 & 0 \\ -1 & -5 & 1 \end{vmatrix} \xrightarrow{EM: R1} (-2)(-4) \cdot - \begin{vmatrix} 4 & 0 \\ -1 & 1 \end{vmatrix} = -8 \cdot 4 = \boxed{-32.}$$

(Note that you can use Sarrus's Method right after Expansion by Minors.)

Using Gaussian Elimination:

$$\begin{aligned} & \begin{vmatrix} -2 & 5 & -4 & -4 \\ 0 & 0 & -4 & 0 \\ 0 & 4 & 2 & 0 \\ 0 & -1 & -5 & 1 \end{vmatrix} \xrightarrow{\textcircled{2} \leftrightarrow \textcircled{4}} (-1) \cdot \begin{vmatrix} -2 & 5 & -4 & -4 \\ 0 & -1 & -5 & 1 \\ 0 & 4 & 2 & 0 \\ 0 & 0 & -4 & 0 \end{vmatrix} \xrightarrow{\textcircled{3} + 4\textcircled{2}} (-1) \cdot \begin{vmatrix} -2 & 5 & -4 & -4 \\ 0 & -1 & -5 & 1 \\ 0 & 0 & -22 & 4 \\ 0 & 0 & -4 & 0 \end{vmatrix} \\ & \xrightarrow{\textcircled{3} \leftrightarrow \textcircled{4}} \begin{vmatrix} -2 & 5 & -4 & -4 \\ 0 & -1 & -5 & 1 \\ 0 & 0 & -4 & 0 \\ 0 & 0 & -22 & 4 \end{vmatrix} \xrightarrow{-\frac{1}{4}\textcircled{3}} -4 \cdot \begin{vmatrix} -2 & 5 & -4 & -4 \\ 0 & -1 & -5 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -22 & 4 \end{vmatrix} \\ & \xrightarrow{\textcircled{4} + 22\textcircled{3}} -4 \cdot \begin{vmatrix} -2 & 5 & -4 & -4 \\ 0 & -1 & -5 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 4 \end{vmatrix} = -4 \cdot (-2 \cdot -1 \cdot 1 \cdot 4) = \boxed{-32.} \end{aligned}$$

Grading for common mistakes: +5 points for using Sarrus's Method right away (on the 4×4 matrix); -5 points for not including the effects of the row operations on the determinant.

4. [20 points] Find the inverse of the matrix $\begin{bmatrix} 3 & -2 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix}$

Solution: Use Gauss-Jordan Elimination:

$$\begin{aligned} \left[\begin{array}{ccc|ccc} 3 & -2 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 & 0 & 1 \end{array} \right] &\xrightarrow{\textcircled{1} + 2\textcircled{3}} \left[\begin{array}{ccc|ccc} 1 & -2 & 1 & 1 & 0 & 2 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 & 0 & 1 \end{array} \right] \\ &\xrightarrow{\textcircled{3} + \textcircled{1}} \left[\begin{array}{ccc|ccc} 1 & -2 & 1 & 1 & 0 & 2 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & -2 & 1 & 1 & 0 & 3 \end{array} \right] \\ &\xrightarrow{\textcircled{3} + 2\textcircled{2}} \left[\begin{array}{ccc|ccc} 1 & -2 & 1 & 1 & 0 & 2 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 2 & 3 \end{array} \right] \\ &\xrightarrow{\textcircled{1} - \textcircled{3}} \left[\begin{array}{ccc|ccc} 1 & -2 & 0 & 0 & -2 & -1 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 2 & 3 \end{array} \right] \\ &\xrightarrow{\textcircled{1} + 2\textcircled{2}} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 2 & 3 \end{array} \right] \end{aligned}$$

Thus, the inverse is $\boxed{\begin{bmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 2 & 3 \end{bmatrix}}$.

Grading: +5 points for the setup, +12 points for the row operations, +3 points for extracting the inverse.

5. [15 points] Use Cramer's Rule to solve the following system of linear equations for (x, y) .

$$\begin{aligned}x + 5y &= -3 \\5x - y &= 0\end{aligned}$$

Solution: Using the notation in class, $A = \begin{bmatrix} 1 & 5 \\ 5 & -1 \end{bmatrix}$, $A_1[B] = \begin{bmatrix} -3 & 5 \\ 0 & -1 \end{bmatrix}$, $A_2[B] = \begin{bmatrix} 1 & -3 \\ 5 & 0 \end{bmatrix}$, and

$$x = \frac{\det A_1[B]}{\det A} = \boxed{\frac{3}{-26}} \quad \text{and} \quad y = \frac{\det A_2[B]}{\det A} = \boxed{\frac{15}{-26}}.$$

Grading for common mistakes: +3 points (total) for another method and a wrong answer; +5 points (total) for another method and a correct answer; full credit was given if Cramer's Rule was used to find the value of one variable, and back substitution for the other.

6. [15 points] Parameterize the solutions to the system of linear equations whose matrix, in reduced row echelon form, is below. Assume that the original variables were x_1, x_2, x_3 , and x_4 .

$$\left[\begin{array}{cccc|c} 1 & 0 & -3 & 1 & -18 \\ 0 & 1 & -1 & -2 & 7 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right]$$

Solution: The variables x_3 and x_4 are the free variables, so let $x_3 = r$ and $x_4 = s$. Then convert the rows of the matrix into equations, solve for the lead variables, and substitute r and s to get:

$$\begin{aligned}x_1 &= -18 + 3x_3 - x_4 = -18 + 3r - s \\x_2 &= 7 + x_3 + 2x_4 = 7 + r + 2s\end{aligned}$$

then collect the equations to get the answer

$\begin{aligned}x_1 &= -18 + 3r - s \\x_2 &= 7 + r + 2s \\x_3 &= r \\x_4 &= s \\ \text{where } r, s &\text{ can be any real numbers}\end{aligned}$
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Grading for common mistakes: -3 points for forgetting "where ... can be any real numbers"; -3 points for missing the equations for the free variables; -2 points for the wrong number of free variables.

242 Test 1 C

SOLUTIONS

1. [15 points] How many solutions does each of the following systems of linear equations have? Circle the entries which led you to your conclusion.

$$(a) \left[\begin{array}{ccc|c} \mathbf{1} & 0 & 0 & -2 \\ 0 & \mathbf{1} & 0 & -2 \\ 0 & 0 & \mathbf{1} & -2 \end{array} \right]$$

Solution: 1

$$(b) \left[\begin{array}{ccc|c} 1 & \mathbf{-5} & 0 & 20 \\ 0 & \mathbf{0} & 1 & -1 \\ 0 & \mathbf{0} & 0 & 0 \\ 0 & \mathbf{0} & 0 & 0 \\ 0 & \mathbf{0} & 0 & 0 \end{array} \right]$$

Solution: ∞

$$(c) \left[\begin{array}{cccc|c} 1 & -4 & -2 & 0 & 3 \\ 0 & 0 & 0 & 1 & 0 \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1} \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right]$$

Solution: 0

Solution: Correct answers are shown above, with the relevant entries in boldface.

Grading: +3 points for the correct answer, +2 points for the work; for each part. Grading for common mistakes: -2 points for “many” (instead of “infinitely many”).

2. [15 points] Solve the system of linear equations

$$-120x_1 + 40x_2 - 15x_3 + 17x_4 = -221$$

$$-30x_1 + 11x_2 - 5x_3 + 5x_4 = -55$$

$$-33x_1 + 10x_2 - 3x_3 + 4x_4 = -61$$

$$-5x_1 + 2x_2 - x_3 + x_4 = -9$$

using the fact that $\begin{bmatrix} -120 & 40 & -15 & 17 \\ -30 & 11 & -5 & 5 \\ -33 & 10 & -3 & 4 \\ -5 & 2 & -1 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & -2 & -2 & 1 \\ 5 & -9 & -10 & 0 \\ 3 & -8 & -5 & 9 \\ -2 & 0 & 5 & 15 \end{bmatrix}$. (Other methods may result in the loss of points.)

Solution:

$$X = A^{-1} \cdot B = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -120 & 40 & -15 & 17 \\ -30 & 11 & -5 & 5 \\ -33 & 10 & -3 & 4 \\ -5 & 2 & -1 & 1 \end{bmatrix}^{-1} \cdot \begin{bmatrix} -221 \\ -55 \\ -61 \\ -9 \end{bmatrix} = \begin{bmatrix} 1 & -2 & -2 & 1 \\ 5 & -9 & -10 & 0 \\ 3 & -8 & -5 & 9 \\ -2 & 0 & 5 & 15 \end{bmatrix} \cdot \begin{bmatrix} -221 \\ -55 \\ -61 \\ -9 \end{bmatrix} = \boxed{\begin{bmatrix} 2 \\ 0 \\ 1 \\ 2 \end{bmatrix}}$$

Grading for common mistakes: +3 points (total) for another method and a wrong answer; +5 points (total) for another method and a correct answer; -5 points for bad matrix multiplication.

3. [20 points] Find $\begin{vmatrix} 2 & 5 & 1 & 0 \\ 0 & -4 & 0 & 0 \\ 3 & -3 & 2 & 2 \\ 4 & 3 & 0 & 1 \end{vmatrix}$

Solution: Typical derivations include the following. Using Expansion by Minors:

$$\begin{vmatrix} 2 & 5 & 1 & 0 \\ 0 & -4 & 0 & 0 \\ 3 & -3 & 2 & 2 \\ 4 & 3 & 0 & 1 \end{vmatrix} \xrightarrow{EM: R2} -4 \cdot \begin{vmatrix} 2 & 1 & 0 \\ 3 & 2 & 2 \\ 4 & 0 & 1 \end{vmatrix} = -4 \cdot [(4) + (8) + (0) - (0) - (0) - (3)] = \boxed{-36.}$$

(Note that you can use Sarrus's Method right after Expansion by Minors.)

Using Gaussian Elimination:

$$\begin{aligned} & \begin{vmatrix} 2 & 5 & 1 & 0 \\ 0 & -4 & 0 & 0 \\ 3 & -3 & 2 & 2 \\ 4 & 3 & 0 & 1 \end{vmatrix} \xrightarrow{\textcircled{1} - \textcircled{3}} \begin{vmatrix} -1 & 8 & -1 & -2 \\ 0 & -4 & 0 & 0 \\ 3 & -3 & 2 & 2 \\ 4 & 3 & 0 & 1 \end{vmatrix} \xrightarrow{\textcircled{3} + 3\textcircled{1}, \textcircled{4} + 4\textcircled{1}} \begin{vmatrix} -1 & 8 & -1 & -2 \\ 0 & -4 & 0 & 0 \\ 0 & 21 & -1 & -4 \\ 0 & 35 & -4 & -7 \end{vmatrix} \\ & \xrightarrow{\textcircled{3} + 5\textcircled{2}} \begin{vmatrix} -1 & 8 & -1 & -2 \\ 0 & -4 & 0 & 0 \\ 0 & 1 & -1 & -4 \\ 0 & 35 & -4 & -7 \end{vmatrix} \xrightarrow{\textcircled{2} \leftrightarrow \textcircled{3}} -1 \cdot \begin{vmatrix} -1 & 8 & -1 & -2 \\ 0 & 1 & -1 & -4 \\ 0 & -4 & 0 & 0 \\ 0 & 35 & -4 & -7 \end{vmatrix} \\ & \xrightarrow{\textcircled{3} + 4\textcircled{2}, \textcircled{4} - 35\textcircled{2}} -1 \cdot \begin{vmatrix} -1 & 8 & -1 & -2 \\ 0 & 1 & -1 & -4 \\ 0 & 0 & -4 & -16 \\ 0 & 0 & 31 & 133 \end{vmatrix} \xrightarrow{\textcircled{4} + 8\textcircled{3}} -1 \cdot \begin{vmatrix} -1 & 8 & -1 & -2 \\ 0 & 1 & -1 & -4 \\ 0 & 0 & -4 & -16 \\ 0 & 0 & -1 & 5 \end{vmatrix} \\ & \xrightarrow{\textcircled{3} \leftrightarrow \textcircled{4}} \begin{vmatrix} -1 & 8 & -1 & -2 \\ 0 & 1 & -1 & -4 \\ 0 & 0 & -1 & 5 \\ 0 & 0 & -4 & -16 \end{vmatrix} \xrightarrow{\textcircled{4} - 4\textcircled{3}} \begin{vmatrix} -1 & 8 & -1 & -2 \\ 0 & 1 & -1 & -4 \\ 0 & 0 & -1 & 5 \\ 0 & 0 & 0 & -36 \end{vmatrix} \\ & = (-1)(1)(-1)(-36) = \boxed{-36.} \end{aligned}$$

Grading for common mistakes: +5 points for using Sarrus's Method right away (on the 4×4 matrix); -5 points for not including the effects of the row operations on the determinant.

4. [20 points] Find the inverse of the matrix $\begin{bmatrix} 0 & 3 & 1 \\ -3 & -1 & 0 \\ -1 & 0 & 0 \end{bmatrix}$

Solution: Use Gauss-Jordan Elimination:

$$\begin{aligned} & \left[\begin{array}{ccc|ccc} 0 & 3 & 1 & 1 & 0 & 0 \\ -3 & -1 & 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 & 0 & 1 \end{array} \right] \xrightarrow{\textcircled{1} - \textcircled{3}} \left[\begin{array}{ccc|ccc} 1 & 3 & 1 & 1 & 0 & -1 \\ -3 & -1 & 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 & 0 & 1 \end{array} \right] \\ & \xrightarrow{\substack{\textcircled{2} + 3\textcircled{1} \\ \textcircled{3} - \textcircled{1}}} \left[\begin{array}{ccc|ccc} 1 & 3 & 1 & 1 & 0 & -1 \\ 0 & 8 & 3 & 3 & 1 & -3 \\ 0 & 3 & 1 & 1 & 0 & 0 \end{array} \right] \\ & \xrightarrow{\textcircled{2} - 3\textcircled{3}} \left[\begin{array}{ccc|ccc} 1 & 3 & 1 & 1 & 0 & -1 \\ 0 & -1 & 0 & 0 & 1 & -3 \\ 0 & 3 & 1 & 1 & 0 & 0 \end{array} \right] \\ & \xrightarrow{-\textcircled{2}} \left[\begin{array}{ccc|ccc} 1 & 3 & 1 & 1 & 0 & -1 \\ 0 & 1 & 0 & 0 & -1 & 3 \\ 0 & 3 & 1 & 1 & 0 & 0 \end{array} \right] \\ & \xrightarrow{\substack{\textcircled{1} - 3\textcircled{2} \\ \textcircled{3} - 3\textcircled{2}}} \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 3 & -10 \\ 0 & 1 & 0 & 0 & -1 & 3 \\ 0 & 0 & 1 & 1 & 3 & -9 \end{array} \right] \\ & \xrightarrow{\textcircled{1} - \textcircled{3}} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & -1 & 3 \\ 0 & 0 & 1 & 1 & 3 & -9 \end{array} \right] \end{aligned}$$

Thus, the inverse is $\boxed{\begin{bmatrix} 0 & 0 & -1 \\ 0 & -1 & 3 \\ 1 & 3 & -9 \end{bmatrix}}$.

Grading: +5 points for the setup, +12 points for the row operations, +3 points for extracting the inverse.

5. [15 points] Use Cramer's Rule to solve the following system of linear equations for (x, y) .

$$\begin{aligned} -2x - 3y &= 4 \\ 4x - y &= 0 \end{aligned}$$

Solution: Using the notation in class, $A = \begin{bmatrix} -2 & -3 \\ 4 & -1 \end{bmatrix}$, $A_1[B] = \begin{bmatrix} 4 & -3 \\ 0 & -1 \end{bmatrix}$, $A_2[B] = \begin{bmatrix} -2 & 4 \\ 4 & 0 \end{bmatrix}$, and

$$x = \frac{\det A_1[B]}{\det A} = \boxed{\frac{-4}{14}} \quad \text{and} \quad y = \frac{\det A_2[B]}{\det A} = \boxed{\frac{-16}{14}}.$$

Grading for common mistakes: +3 points (total) for another method and a wrong answer; +5 points (total) for another method and a correct answer; full credit was given if Cramer's Rule was used to find the value of one variable, and back substitution for the other.

6. [15 points] Parameterize the solutions to the system of linear equations whose matrix, in reduced row echelon form, is below. Assume that the original variables were x_1, x_2, x_3, x_4, x_5 , and x_6 .

$$\left[\begin{array}{cccccc|c} 1 & -4 & 5 & 0 & 0 & 3 & 5 \\ 0 & 0 & 0 & 1 & 0 & 3 & 6 \\ 0 & 0 & 0 & 0 & 1 & 4 & 8 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array} \right]$$

Solution: The variables x_2, x_3 , and x_6 are the free variables, so let $x_2 = r$, $x_3 = s$, and $x_6 = t$. Then convert the rows of the matrix into equations, solve for the lead variables, and substitute r and s to get:

$$\begin{aligned} x_1 &= 5 + 4x_2 - 5x_3 - 3x_6 = 5 + 4r - 5s - 3t \\ x_4 &= 6 - 3x_6 = 6 - 3t \\ x_5 &= 8 - 4x_6 = 8 - 4t \end{aligned}$$

then collect the equations to get the answer

$$\boxed{\begin{aligned} x_1 &= 5 + 4r - 5s - 3t \\ x_2 &= r \\ x_3 &= s \\ x_4 &= 6 - 3t \\ x_5 &= 8 - 4t \\ x_6 &= t \\ \text{where } r, s, t &\text{ can be any real numbers} \end{aligned}}$$

Grading for common mistakes: -3 points for forgetting "where ... can be any real numbers"; -3 points for missing the equations for the free variables; -2 points for the wrong number of free variables.

242 Test 1 D

SOLUTIONS

1. [15 points] How many solutions does each of the following systems of linear equations have? Circle the entries which led you to your conclusion.

$$(a) \left[\begin{array}{cccc|c} \mathbf{1} & 0 & 0 & 0 & 4 \\ 0 & \mathbf{1} & 0 & 0 & 3 \\ 0 & 0 & \mathbf{1} & 0 & -4 \\ 0 & 0 & 0 & \mathbf{1} & -3 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right]$$

Solution: 1

$$(b) \left[\begin{array}{ccccc|c} 1 & 5 & 0 & 0 & 0 & 5 \\ 0 & 0 & 1 & 0 & 0 & 5 \\ 0 & 0 & 0 & 1 & 0 & 4 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & -2 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array} \right]$$

Solution: 0

$$(c) \left[\begin{array}{cccc|c} 1 & \mathbf{-4} & 0 & \mathbf{2} & 0 & 23 \\ 0 & \mathbf{0} & 1 & \mathbf{2} & 0 & -4 \\ 0 & \mathbf{0} & 0 & \mathbf{0} & 1 & 1 \\ 0 & \mathbf{0} & 0 & \mathbf{0} & 0 & 0 \end{array} \right]$$

Solution: ∞

Solution: Correct answers are shown above, with the relevant entries in boldface.

Grading: +3 points for the correct answer, +2 points for the work; for each part. Grading for common mistakes: -2 points for “many” (instead of “infinitely many”).

2. [15 points] Solve the system of linear equations

$$\begin{aligned} 145x_1 - 350x_2 + 132x_3 + 58x_4 &= 538 \\ -42x_1 + 104x_2 - 39x_3 - 17x_4 &= -157 \\ -7x_1 + 19x_2 - 7x_3 - 3x_4 &= -27 \\ 3x_1 - 5x_2 + 2x_3 + x_4 &= 10 \end{aligned}$$

using the fact that $\left[\begin{array}{cccc} 145 & -350 & 132 & 58 \\ -42 & 104 & -39 & -17 \\ -7 & 19 & -7 & -3 \\ 3 & -5 & 2 & 1 \end{array} \right]^{-1} = \left[\begin{array}{cccc} -1 & -4 & 4 & 2 \\ 1 & 3 & 1 & -4 \\ 2 & 4 & 11 & -15 \\ 4 & 19 & -29 & 5 \end{array} \right]$. (Other methods may result in the loss of points.)

Solution:

$$X = A^{-1} \cdot B = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 145 & -350 & 132 & 58 \\ -42 & 104 & -39 & -17 \\ -7 & 19 & -7 & -3 \\ 3 & -5 & 2 & 1 \end{bmatrix}^{-1} \cdot \begin{bmatrix} 538 \\ -157 \\ -27 \\ 10 \end{bmatrix} = \begin{bmatrix} -1 & -4 & 4 & 2 \\ 1 & 3 & 1 & -4 \\ 2 & 4 & 11 & -15 \\ 4 & 19 & -29 & 5 \end{bmatrix} \cdot \begin{bmatrix} 538 \\ -157 \\ -27 \\ 10 \end{bmatrix} = \boxed{\begin{bmatrix} 2 \\ 0 \\ 1 \\ 2 \end{bmatrix}}$$

Grading for common mistakes: +3 points (total) for another method and a wrong answer; +5 points (total) for another method and a correct answer; -5 points for bad matrix multiplication.

3. [20 points] Find $\begin{vmatrix} 0 & 5 & 0 & 4 \\ 3 & -4 & 0 & -3 \\ -5 & -1 & 0 & -5 \\ -1 & -2 & -4 & -5 \end{vmatrix}$

Solution: Typical derivations include the following. Using Expansion by Minors:

$$\begin{vmatrix} 0 & 5 & 0 & 4 \\ 3 & -4 & 0 & -3 \\ -5 & -1 & 0 & -5 \\ -1 & -2 & -4 & -5 \end{vmatrix} \xrightarrow{EM : C3} (-4) \cdot \begin{vmatrix} 0 & 5 & 4 \\ 3 & -4 & -3 \\ -5 & -1 & -5 \end{vmatrix} = (-4) \cdot -[0 + (75) + (-12) - (80) - (0) - (-75)] = \boxed{232.}$$

(Note that you can use Sarrus's Method right after Expansion by Minors.)

Using Gaussian Elimination:

$$\begin{vmatrix} 0 & 5 & 0 & 4 \\ 3 & -4 & 0 & -3 \\ -5 & -1 & 0 & -5 \\ -1 & -2 & -4 & -5 \end{vmatrix} \xrightarrow{\textcircled{1} - \textcircled{4}} \begin{vmatrix} 1 & 7 & 4 & 9 \\ 3 & -4 & 0 & -3 \\ -5 & -1 & 0 & -5 \\ -1 & -2 & -4 & -5 \end{vmatrix} \xrightarrow{\begin{matrix} \textcircled{2} - 3\textcircled{1} \\ \textcircled{3} + 5\textcircled{1} \\ \textcircled{4} + \textcircled{1} \end{matrix}} \begin{vmatrix} 1 & 7 & 4 & 9 \\ 0 & -25 & -12 & -30 \\ 0 & 34 & 20 & 40 \\ 0 & 5 & 0 & 4 \end{vmatrix} \\ \xrightarrow{\textcircled{2} \leftrightarrow \textcircled{3}} -1 \cdot \begin{vmatrix} 1 & 7 & 4 & 9 \\ 0 & 34 & 20 & 40 \\ 0 & -25 & -12 & -30 \\ 0 & 5 & 0 & 4 \end{vmatrix} \xrightarrow{\begin{matrix} \textcircled{2} - 7\textcircled{4} \\ \textcircled{3} + 5\textcircled{4} \end{matrix}} -1 \cdot \begin{vmatrix} 1 & 7 & 4 & 9 \\ 0 & -1 & 20 & 12 \\ 0 & 0 & -12 & -10 \\ 0 & 5 & 0 & 4 \end{vmatrix} \\ \xrightarrow{\textcircled{4} + 5\textcircled{2}} -1 \cdot \begin{vmatrix} 1 & 7 & 4 & 9 \\ 0 & -1 & 20 & 12 \\ 0 & 0 & -12 & -10 \\ 0 & 0 & 100 & 64 \end{vmatrix} \xrightarrow{\textcircled{4} + 8\textcircled{3}} -1 \cdot \begin{vmatrix} 1 & 7 & 4 & 9 \\ 0 & -1 & 20 & 12 \\ 0 & 0 & -12 & -10 \\ 0 & 0 & 4 & -16 \end{vmatrix} \\ \xrightarrow{\textcircled{3} \leftrightarrow \textcircled{4}} \begin{vmatrix} 1 & 7 & 4 & 9 \\ 0 & -1 & 20 & 12 \\ 0 & 0 & 4 & -16 \\ 0 & 0 & -12 & -10 \end{vmatrix} \xrightarrow{\textcircled{4} + 3\textcircled{3}} \begin{vmatrix} 1 & 7 & 4 & 9 \\ 0 & -1 & 20 & 12 \\ 0 & 0 & 4 & -16 \\ 0 & 0 & 0 & -58 \end{vmatrix} \\ = 1 \cdot (-1) \cdot 4 \cdot (-58) = \boxed{-232.}$$

Grading for common mistakes: +5 points for using Sarrus's Method right away (on the 4×4 matrix); -5 points for not including the effects of the row operations on the determinant.

4. [20 points] Find the inverse of the matrix $\begin{bmatrix} 8 & -2 & 1 \\ 1 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix}$

Solution: Use Gauss-Jordan Elimination:

$$\begin{aligned} \left[\begin{array}{ccc|ccc} 8 & -2 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 & 0 & 1 \end{array} \right] &\xrightarrow{\textcircled{1} \leftrightarrow \textcircled{2}} \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 0 & 1 & 0 \\ 8 & -2 & 1 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 1 \end{array} \right] \\ &\xrightarrow{\substack{\textcircled{2} - 8\textcircled{1} \\ \textcircled{3} + \textcircled{1}}} \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & -10 & 1 & 1 & -8 & 0 \\ 0 & 1 & 0 & 0 & 1 & 1 \end{array} \right] \\ &\xrightarrow{\textcircled{2} \leftrightarrow \textcircled{3}} \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & -10 & 1 & 1 & -8 & 0 \end{array} \right] \\ &\xrightarrow{\substack{\textcircled{3} + 10\textcircled{2} \\ \textcircled{1} - \textcircled{2}}} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 2 & 10 \end{array} \right] \end{aligned}$$

Thus, the inverse is $\boxed{\begin{bmatrix} 0 & 0 & -1 \\ 0 & 1 & 1 \\ 1 & 2 & 10 \end{bmatrix}}$.

Grading: +5 points for the setup, +12 points for the row operations, +3 points for extracting the inverse.

5. [15 points] Use Cramer's Rule to solve the following system of linear equations for (x, y) .

$$\begin{aligned} -x - 2y &= -1 \\ 5x - 3y &= 1 \end{aligned}$$

Solution: Using the notation in class, $A = \begin{bmatrix} -1 & -2 \\ 5 & -3 \end{bmatrix}$, $A_1[B] = \begin{bmatrix} -1 & -2 \\ 1 & -3 \end{bmatrix}$, $A_2[B] = \begin{bmatrix} -1 & -1 \\ 5 & 1 \end{bmatrix}$, and

$$x = \frac{\det A_1[B]}{\det A} = \boxed{\frac{5}{13}} \quad \text{and} \quad y = \frac{\det A_2[B]}{\det A} = \boxed{\frac{4}{13}}.$$

Grading for common mistakes: +3 points (total) for another method and a wrong answer; +5 points (total) for another method and a correct answer; full credit was given if Cramer's Rule was used to find the value of one variable, and back substitution for the other.

6. [15 points] Parameterize the solutions to the system of linear equations whose matrix, in reduced row echelon form, is below. Assume that the original variables were x_1, x_2, x_3, x_4 , and x_5 .

$$\left[\begin{array}{ccccc|c} 1 & 0 & -2 & 0 & 0 & 1 \\ 0 & 1 & 5 & 0 & 0 & 6 \\ 0 & 0 & 0 & 1 & 0 & -5 \\ 0 & 0 & 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array} \right]$$

Solution: The variable x_3 is the only free variable, so let $x_3 = r$. Then convert the rows of the matrix into equations, solve for the lead variables, and substitute r and s to get:

$$\begin{aligned} x_1 &= 1 + 2x_3 = 1 + 2r \\ x_2 &= 6 - 5x_3 = 6 - 5r \\ x_4 &= -5 \\ x_5 &= -3 \end{aligned}$$

then collect the equations to get the answer

$\begin{aligned} x_1 &= 1 + 2r \\ x_2 &= 6 - 5r \\ x_3 &= r \\ x_4 &= -5 \\ x_5 &= -3 \\ \text{where } r &\text{ can be any real number} \end{aligned}$

Grading for common mistakes: -3 points for forgetting "where ... can be any real numbers"; -3 points for missing the equations for the free variables; -2 points for the wrong number of free variables.