

C. HECKMAN SOLUTIONS

242 TEST 1C

- (1) [15 points] Parameterize the solutions to the system of linear equations whose matrix, in reduced row echelon form, is below. Assume that the original variables were x_1, x_2, x_3, \dots

$$\left[\begin{array}{cccc|c} 1 & 2 & 0 & -5 & -3 \\ 0 & 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right]$$

Solution: The free variables are x_2 and x_4 , since there are no pivots in their columns. They are set equal to parameters: $x_2 = s$ and $x_4 = t$, for instance. Then the rows of the matrix are turned into equations:

$$\begin{array}{rcl} x_1 + 2x_2 - 5x_4 = -3 & \implies & x_1 = -3 - 2x_2 + 5x_4 = -3 - 2s + 5t \\ x_3 - 2x_4 = 1 & & x_3 = 1 + 2x_4 = 1 + 2t \end{array}$$

so the solution is

$$\begin{array}{l} x_1 = -3 - 2s + 5t \\ x_2 = s \\ x_3 = 1 + 2t \\ x_4 = t \\ \text{where } s \text{ and } t \text{ are any real numbers.} \end{array}$$

Grading; +5 points for assigning parameters, +5 points for finding the equations, +2 points for combining equations, +3 points for “ s and t can be any real numbers.”

- (2) [15 points] Use Cramer’s Rule to find the value of x in the system

$$\begin{array}{r} x - 5y = 8 \\ 6x + 8y = 1 \end{array}$$

Solution: This system is of the form $AX = B$, where $A = \begin{bmatrix} 1 & -5 \\ 6 & 8 \end{bmatrix}$. $A_1[B] = \begin{bmatrix} 8 & -5 \\ 1 & 8 \end{bmatrix}$, and

$$x = \frac{\det A_1[B]}{\det A} = \frac{8 \cdot 8 - (-5) \cdot 1}{1 \cdot 8 - (-5) \cdot 6} = \boxed{\frac{69}{38}}$$

Grading: +5 points for each determinant, +5 points for Cramer’s Rule. *Grading for common mistakes:* +10 points (total) for an alternate method, done correctly.

- (3) [15 points] How many solutions does each of the following systems of linear equations have? Circle the entries which led you to your conclusion.

$$(a) \left[\begin{array}{cccc|c} 1 & 0 & -5 & -5 & 2 \\ 0 & 1 & -2 & 4 & 4 \\ \boxed{0} & \boxed{0} & \boxed{0} & \boxed{0} & \boxed{2} \end{array} \right]$$

Solution: None.

$$(b) \left[\begin{array}{cccc|c} 1 & \boxed{3} & 0 & \boxed{5} & -1 \\ 0 & \boxed{0} & 1 & \boxed{-4} & 3 \\ 0 & \boxed{0} & 0 & \boxed{0} & 0 \end{array} \right]$$

Solution: Infinitely many.

$$(c) \left[\begin{array}{ccc|c} \boxed{1} & 0 & 0 & 3 \\ 0 & \boxed{1} & 0 & 4 \\ 0 & 0 & \boxed{1} & 2 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

Solution: Exactly one.

Solution: Correct answers, and the appropriate entries, are shown above. Grading: +4 points (each) for correct answer but incorrect work; +2 points (each) for an incorrect answer but supporting work; +1 point (each) for incorrect answers with no work.

- (4) [20 points] Find the inverse of the matrix $\begin{bmatrix} 1 & 0 & 2 \\ -4 & 1 & -11 \\ 2 & 0 & 3 \end{bmatrix}$

Solution:

$$\begin{aligned} \left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ -4 & 1 & -11 & 0 & 1 & 0 \\ 2 & 0 & 3 & 0 & 0 & 1 \end{array} \right] & \xrightarrow{\substack{\textcircled{2} + 4\textcircled{1} \\ \textcircled{3} - 2\textcircled{2}}} \left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ 0 & 1 & -3 & 4 & 1 & 0 \\ 0 & 0 & -1 & -2 & 0 & 1 \end{array} \right] \\ & \xrightarrow{-\textcircled{3}} \left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ 0 & 1 & -3 & 4 & 1 & 0 \\ 0 & 0 & 1 & 2 & 0 & -1 \end{array} \right] \\ & \xrightarrow{\substack{\textcircled{1} - 2\textcircled{3} \\ \textcircled{2} + 3\textcircled{3}}} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -3 & 0 & 2 \\ 0 & 1 & 0 & 10 & 1 & -3 \\ 0 & 0 & 1 & 2 & 0 & -1 \end{array} \right] \end{aligned}$$

The inverse of $\begin{bmatrix} 1 & 0 & 2 \\ -4 & 1 & -11 \\ 2 & 0 & 3 \end{bmatrix}$ is thus $\boxed{\begin{bmatrix} -3 & 0 & 2 \\ 10 & 1 & -3 \\ 2 & 0 & -1 \end{bmatrix}}$.

Grading: +5 points for the setup, +10 points for Gauss-Jordan Elimination, +5 points for indicating what the matrix is.

(5) [15 points] Solve the system of linear equations

$$\begin{aligned}x + 3y - 3z &= -5 \\x + 2y - 3z &= -24 \\-3x - 6y + 8z &= 51\end{aligned}$$

using the fact that $\begin{bmatrix} 1 & 3 & -3 \\ 1 & 2 & -3 \\ -3 & -6 & 8 \end{bmatrix}^{-1} = \begin{bmatrix} -2 & -6 & -3 \\ 1 & -1 & 0 \\ 0 & -3 & -1 \end{bmatrix}$

Solution: This system can be written as $AX = B$, where $A = \begin{bmatrix} 1 & 3 & -3 \\ 1 & 2 & -3 \\ -3 & -6 & 8 \end{bmatrix}$ and

$$B = \begin{bmatrix} -5 \\ -24 \\ 51 \end{bmatrix}. \text{ Then}$$

$$X = A^{-1} \cdot B = \begin{bmatrix} -2 & -6 & -3 \\ 1 & -1 & 0 \\ 0 & -3 & -1 \end{bmatrix} \cdot \begin{bmatrix} -5 \\ -24 \\ 51 \end{bmatrix} = \boxed{\begin{bmatrix} 1 \\ 19 \\ 21 \end{bmatrix}}.$$

Grading: +5 points for the $A^{-1} \cdot B$ formula, +5 points for substitution, +5 points for the matrix arithmetic.

(6) [20 points] Find $\begin{vmatrix} 3 & 1 & 4 & -2 \\ 0 & 0 & 2 & 0 \\ 2 & 5 & -1 & 1 \\ 0 & 0 & 0 & 4 \end{vmatrix}$

Solution: There are several ways to calculate this determinant; two are shown below.

$$\begin{aligned} \begin{vmatrix} 3 & 1 & 4 & -2 \\ 0 & 0 & 2 & 0 \\ 2 & 5 & -1 & 1 \\ 0 & 0 & 0 & 4 \end{vmatrix} &\xrightarrow{\text{EM : R2}} (-1) \cdot 2 \cdot \begin{vmatrix} 3 & 1 & -2 \\ 2 & 5 & 1 \\ 0 & 0 & 4 \end{vmatrix} \xrightarrow{\text{EM : R3}} (-2) \cdot 4 \cdot (+1) \cdot \begin{vmatrix} 3 & 1 \\ 2 & 5 \end{vmatrix} \\ &= (-8) \cdot (3 \cdot 5 - 1 \cdot 2) = \boxed{-104}. \end{aligned}$$

(Here, EM : Ri denotes expansion by minors along row i.)

\Rightarrow

$$\begin{aligned}
& \begin{vmatrix} 3 & 1 & 4 & -2 \\ 0 & 0 & 2 & 0 \\ 2 & 5 & -1 & 1 \\ 0 & 0 & 0 & 4 \end{vmatrix} \xrightarrow{\textcircled{1} - 2\textcircled{3}} \begin{vmatrix} 1 & -4 & 5 & -3 \\ 0 & 0 & 2 & 0 \\ 2 & 5 & -1 & 1 \\ 0 & 0 & 0 & 4 \end{vmatrix} \xrightarrow{\textcircled{3} - 2\textcircled{1}} \begin{vmatrix} 1 & -4 & 5 & -3 \\ 0 & 0 & 2 & 0 \\ 0 & 13 & -11 & 7 \\ 0 & 0 & 0 & 4 \end{vmatrix} \\
& \xrightarrow{\textcircled{2} \leftrightarrow \textcircled{3}} (-1) \cdot \begin{vmatrix} 1 & -4 & 5 & -3 \\ 0 & 13 & -11 & 7 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 4 \end{vmatrix} = (-1) \cdot 1 \cdot 13 \cdot 2 \cdot 4 = \boxed{-104}.
\end{aligned}$$

Grading for common mistakes: +10 points (total) for an illegal row operation; +15 points (total) for legal row operations but bad planning; -3 points for a row operation that changes the determinant (like $2\textcircled{1} + \textcircled{3}$, which is the same as $R_1 = 2r_1 + r_3$); -5 points for $a \cdot \det A = \det(aA)$.

C. HECKMAN SOLUTIONS

242 TEST 1B

- (1) [15 points] Parameterize the solutions to the system of linear equations whose matrix, in reduced row echelon form, is below. Assume that the original variables were x_1, x_2, x_3, \dots

$$\left[\begin{array}{cccc|c} 1 & 3 & 0 & 5 & -1 \\ 0 & 0 & 1 & -4 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right]$$

Solution: The free variables are x_2 and x_4 , since there are no pivots in their columns. They are set equal to parameters: $x_2 = s$ and $x_4 = t$, for instance. Then the rows of the matrix are turned into equations:

$$\begin{array}{rcl} x_1 + 3x_2 + 5x_4 = -1 & \implies & x_1 = -3 - 3x_2 - 5x_4 = -3 - 3s - 5t \\ x_3 - 4x_4 = 3 & & x_3 = 3 + 4x_4 = 3 + 4t \end{array}$$

so the solution is

$$\begin{array}{l} x_1 = -3 - 3s - 5t \\ x_2 = s \\ x_3 = 3 + 4t \\ x_4 = t \\ \text{where } s \text{ and } t \text{ are any real numbers.} \end{array}$$

Grading; +5 points for assigning parameters, +5 points for finding the equations, +2 points for combining equations, +3 points for “ s and t can be any real numbers.”

- (2) [15 points] Use Cramer’s Rule to find the value of y in the system

$$\begin{array}{r} 4x + 8y = 6 \\ -5x + 4y = 7 \end{array}$$

Solution: This system is of the form $AX = B$, where $A = \begin{bmatrix} 4 & 8 \\ -5 & 4 \end{bmatrix}$. $A_1[B] =$

$$\begin{bmatrix} 4 & 6 \\ -5 & 7 \end{bmatrix}, \text{ and}$$

$$x = \frac{\det A_1[B]}{\det A} = \frac{4 \cdot 7 - 6 \cdot (-5)}{4 \cdot 4 - 8 \cdot (-5)} = \boxed{\frac{58}{56}}.$$

Grading: +5 points for each determinant, +5 points for Cramer’s Rule. *Grading for common mistakes:* +10 points (total) for an alternate method, done correctly.

- (3) [15 points] How many solutions does each of the following systems of linear equations have? Circle the entries which led you to your conclusion.

$$(a) \left[\begin{array}{ccc|c} 1 & 0 & -1 & 5 \\ 0 & 1 & 3 & -2 \\ 0 & 0 & 0 & 1 \end{array} \right]$$

Solution: Infinitely many.

$$(b) \left[\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & -3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array} \right]$$

Solution: Exactly one.

$$(c) \left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -4 \\ 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 1 \end{array} \right]$$

Solution: None.

Solution: Correct answers, and the appropriate entries, are shown above. Grading: +4 points (each) for correct answer but incorrect work; +2 points (each) for an incorrect answer but supporting work; +1 point (each) for incorrect answers with no work.

- (4) [20 points] Find the inverse of the matrix $\begin{bmatrix} 1 & 2 & 4 \\ -2 & -3 & -8 \\ 2 & 4 & 7 \end{bmatrix}$

Solution:

$$\begin{aligned} \left[\begin{array}{ccc|ccc} 1 & 2 & 4 & 1 & 0 & 0 \\ -2 & -3 & -8 & 0 & 1 & 0 \\ 2 & 4 & 7 & 0 & 0 & 1 \end{array} \right] & \xrightarrow{\substack{\textcircled{2} + 2\textcircled{1} \\ \textcircled{3} - 2\textcircled{1}}} \left[\begin{array}{ccc|ccc} 1 & 2 & 4 & 1 & 0 & 0 \\ 0 & 1 & 0 & 2 & 1 & 0 \\ 0 & 0 & -1 & -2 & 0 & 1 \end{array} \right] \\ & \xrightarrow{-\textcircled{3}} \left[\begin{array}{ccc|ccc} 1 & 2 & 4 & 1 & 0 & 0 \\ 0 & 1 & 0 & 2 & 1 & 0 \\ 0 & 0 & 1 & 2 & 0 & -1 \end{array} \right] \\ & \xrightarrow{\textcircled{1} - 4\textcircled{3}} \left[\begin{array}{ccc|ccc} 1 & 2 & 0 & -7 & 0 & 4 \\ 0 & 1 & 0 & 2 & 1 & 0 \\ 0 & 0 & 1 & 2 & 0 & -1 \end{array} \right] \\ & \xrightarrow{\textcircled{1} - 2\textcircled{2}} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -11 & 2 & -4 \\ 0 & 1 & 0 & 2 & 1 & 0 \\ 0 & 0 & 1 & 2 & 0 & -1 \end{array} \right] \end{aligned}$$

\Rightarrow

The inverse of $\begin{bmatrix} 1 & 2 & 4 \\ -2 & -3 & -8 \\ 2 & 4 & 7 \end{bmatrix}$ is thus $\boxed{\begin{bmatrix} -11 & 2 & -4 \\ 2 & 1 & 0 \\ 2 & 0 & -1 \end{bmatrix}}$.

Grading: +5 points for the setup, +10 points for Gauss-Jordan Elimination, +5 points for indicating what the matrix is.

(5) [15 points] Solve the system of linear equations

$$\begin{aligned} -x - 3y + z &= -37 \\ -x - 2y - 2z &= -81 \\ 3x + 8y - z &= 134 \end{aligned}$$

using the fact that $\begin{bmatrix} -1 & -3 & 1 \\ -1 & -2 & -2 \\ 3 & 8 & -1 \end{bmatrix}^{-1} = \begin{bmatrix} 18 & 5 & 8 \\ -7 & -2 & -3 \\ -2 & -1 & -1 \end{bmatrix}$

Solution: This system can be written as $AX = B$, where $A = \begin{bmatrix} -1 & -3 & 1 \\ -1 & -2 & -2 \\ 3 & 8 & -1 \end{bmatrix}$ and

$$B = \begin{bmatrix} -37 \\ -81 \\ 134 \end{bmatrix}. \text{ Then}$$

$$X = A^{-1} \cdot B = \begin{bmatrix} 18 & 5 & 8 \\ -7 & -2 & -3 \\ -2 & -1 & -1 \end{bmatrix} \cdot \begin{bmatrix} -37 \\ -81 \\ 134 \end{bmatrix} = \boxed{\begin{bmatrix} 1 \\ 19 \\ 21 \end{bmatrix}}.$$

Grading: +5 points for the $A^{-1} \cdot B$ formula, +5 points for substitution, +5 points for the matrix arithmetic.

(6) [20 points] Find $\begin{vmatrix} 1 & 0 & 6 & 0 \\ 5 & 0 & 3 & -4 \\ 2 & -3 & -1 & 7 \\ 1 & 0 & 5 & 0 \end{vmatrix}$

Solution: There are several ways to calculate this determinant; two are shown below.

$$\begin{aligned} \begin{vmatrix} 1 & 0 & 6 & 0 \\ 5 & 0 & 3 & -4 \\ 2 & -3 & -1 & 7 \\ 1 & 0 & 5 & 0 \end{vmatrix} &\xrightarrow{\text{EM : R3}} (-3) \cdot (-1) \cdot \begin{vmatrix} 1 & 6 & 0 \\ 5 & 3 & -4 \\ 1 & 5 & 0 \end{vmatrix} \xrightarrow{\text{EM : C3}} 3 \cdot (-4) \cdot (-1) \cdot \begin{vmatrix} 1 & 6 \\ 1 & 5 \end{vmatrix} \\ &= 3 \cdot 4 \cdot (1 \cdot 5 - 6 \cdot 1) = \boxed{-12}. \end{aligned}$$

(Here, EM : Ri denotes expansion by minors along row i, and EM : Cj denotes expansion by minors along column j.)

⇒

$$\begin{aligned}
& \begin{vmatrix} 1 & 0 & 6 & 0 \\ 5 & 0 & 3 & -4 \\ 2 & -3 & -1 & 7 \\ 1 & 0 & 5 & 0 \end{vmatrix} \xrightarrow{\substack{\textcircled{2} - 5\textcircled{1} \\ \textcircled{3} - 2\textcircled{1} \\ \textcircled{4} - \textcircled{1}}} \begin{vmatrix} 1 & 0 & 6 & 0 \\ 0 & 0 & -27 & -4 \\ 0 & -3 & -13 & 7 \\ 0 & 0 & -1 & 0 \end{vmatrix} \xrightarrow{\substack{\textcircled{2} \leftrightarrow \textcircled{3}}} (-1) \cdot \begin{vmatrix} 1 & 0 & 6 & 0 \\ 0 & -3 & -13 & 7 \\ 0 & 0 & -27 & -4 \\ 0 & 0 & -1 & 0 \end{vmatrix} \\
& \xrightarrow{\substack{\textcircled{3} - 28\textcircled{4}}} (-1) \cdot \begin{vmatrix} 1 & 0 & 6 & 0 \\ 0 & -3 & -13 & 7 \\ 0 & 0 & 1 & -4 \\ 0 & 0 & -1 & 0 \end{vmatrix} \\
& \xrightarrow{\substack{\textcircled{4} + \textcircled{3}}} (-1) \cdot \begin{vmatrix} 1 & 0 & 6 & 0 \\ 0 & -3 & -13 & 7 \\ 0 & 0 & 1 & -4 \\ 0 & 0 & 0 & -4 \end{vmatrix} \\
& = (-1) \cdot (1 \cdot (-3) \cdot 1 \cdot (-4)) = \boxed{-12.}
\end{aligned}$$

Grading for common mistakes: +10 points (total) for an illegal row operation; +15 points (total) for legal row operations but bad planning; -3 points for a row operation that changes the determinant (like $2\textcircled{1} + \textcircled{3}$, which is the same as $R_1 = 2r_1 + r_3$); -5 points for $a \cdot \det A = \det(aA)$.