7.3 Coordinates and Change of Basis

If $B = \{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k\}$ is a basis for a subspace W, then every vector $\vec{u}$ in W can be written in the form

$$c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_k\vec{v}_k$$

because B spans W.

At the same time, we cannot have two different linear combinations of the $\vec{v}_i$'s that add up to $\vec{u}$, because we could subtract one from the other and get a non-trivial linear combination of the $\vec{v}_i$'s that adds up to $\vec{0}$; this is because B is linearly independent.

So this representation of $\vec{u}$ is unique!

If we arrange the vectors in a basis in a certain order, we can put these numbers c_i into a vector. This vector of the c_i's is called the coordinate vector of $\vec{u}$ with respect to the ordered basis B and is sometimes denoted $[\vec{u}]_B$.

When the order of the vectors in a basis matters, the basis is often written using parentheses: $(\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k)$. This is called an ordered basis.

Finding the coordinates of a vector $\vec{u}$ with respect to a basis B is just like determining whether $\vec{u}$ is in the span of B, except that there will be exactly one solution when it is! So we can re-use our procedure from Chapter 4 to find the coordinates of a basis, and we can even use inverses to write it in a purely algebraic manner.

Example 1. Find the coordinates of $\vec{u} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ with respect to the ordered basis

$$B = \left(\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -3 \\ 2 \end{bmatrix} \right).$$

$$\begin{bmatrix} 1 & 0 & 1 \\ -2 & 1 & -3 \\ 1 & 1 & 2 \end{bmatrix} \cdot \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

$$\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ -2 & 1 & -3 \\ 1 & 1 & 2 \end{bmatrix}^{-1} \cdot \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

$$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}_B = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 5/2 & 1/2 & -1/2 \\ 1/2 & 1/2 & 1/2 \\ -3/2 & -1/2 & 1/2 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \boxed{\begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix}}$$

In short, if $\widetilde{B}$ is the matrix obtained by "gluing" the vectors of B together, then the coordinate vector of $\vec{u}$ with respect to B is

$$\boxed{[\vec{u}]_B = \widetilde{B}^{-1} \cdot \vec{u}}$$

This helps us, because we might have two bases (B and C) for a vector space, and we want to convert from coordinates in one basis into coordinates in the other basis.

The equation $[\vec{u}]_B = \widetilde{B}^{-1} \cdot \vec{u}$ **tells us that** $\vec{u} = \widetilde{B} \cdot [\vec{u}]_B$**. We use this equation twice, one for each basis:**

$$\vec{u} = \widetilde{B} \cdot [\vec{u}]_B \quad \textbf{and} \quad \vec{u} = \widetilde{C} \cdot [\vec{u}]_C.$$

Since the vector $\vec{u}$ **is the same in both cases, we can combine these equations and solve for** $[\vec{u}]_C$**:**

$$\boxed{[\vec{u}]_C = \widetilde{C}^{-1} \cdot \widetilde{B} \cdot [\vec{u}]_B}$$

and now we can just do some number crunching!

Incidentally, the matrix $\widetilde{C}^{-1} \cdot \widetilde{B}$ **is sometimes called the change of basis matrix.**

Example 2. If the coordinates of $\vec{u}$ with respect to the ordered basis $B = \left(\begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}, \begin{bmatrix} -1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ -2 \end{bmatrix} \right)$ are $\begin{bmatrix} 2 \\ 4 \\ 2 \end{bmatrix}$, what are the coordinates of $\vec{u}$ with respect to the ordered basis $C = \left(\begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ -1 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ 0 \\ -2 \end{bmatrix} \right)$?

Example 2. If the coordinates of $\vec{u}$ with respect to the ordered basis $B = \left(\begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}, \begin{bmatrix} -1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ -2 \end{bmatrix} \right)$ are $\begin{bmatrix} 2 \\ 4 \\ 2 \end{bmatrix}$, what are the coordinates of $\vec{u}$ with respect to the ordered basis $C = \left(\begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ -1 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ 0 \\ -2 \end{bmatrix} \right)$?

We know $\widetilde{B} = \begin{bmatrix} 2 & -1 & 1 \\ 3 & -1 & 3 \\ 4 & 0 & -2 \end{bmatrix}$, $\widetilde{C} = \begin{bmatrix} 2 & 3 & 4 \\ -1 & -1 & 0 \\ 1 & 3 & -2 \end{bmatrix}$, **and**

$[\vec{u}]_B = \begin{bmatrix} 2 \\ 4 \\ 2 \end{bmatrix}$.

Our formula (a few slides back) tells us that

$$[\vec{u}]_C = \widetilde{C}^{-1} \cdot \widetilde{B} \cdot [\vec{u}]_B$$

$$[\vec{u}]_C = \begin{bmatrix} 2 & 3 & 4 \\ -1 & -1 & 0 \\ 1 & 3 & -2 \end{bmatrix}^{-1} \cdot \begin{bmatrix} 2 & -1 & 1 \\ 3 & -1 & 3 \\ 4 & 0 & -2 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 4 \\ 2 \end{bmatrix}$$

$$[\vec{u}]_C = \boxed{\begin{bmatrix} -82/5 \\ 42/5 \\ 12/5 \end{bmatrix}}$$

$\square$