6.2. Diagonalization

Recall that, when you have a polynomial in factored form

$$(x - r_1)^{e_1}(x - r_2)^{e_2} \cdots (x - r_k)^{e_k}$$

with the r_i's all distinct, that the multiplicity of r_i is said to be e_i.

In Example 1 from the previous section, A had two distinct eigenvalues (with multiplicity 1 in the characteristic equation), and each had an eigenspace of dimension 1. In Example 2, B had two distinct eigenvalues (with multiplicity 2), one of which had an eigenspace of dimension 1, and the other eigenvalue had an eigenspace of dimension 2.

So the dimension of the eigenspace can be less than or equal to the multiplicity of the eigenvalue. Surprisingly, the dimension of the eigenspace cannot be greater than the multiplicity of the eigenvalue in the characteristic equation.

A matrix A where the dimension of the eigenspace equals the multiplicity of the corresponding eigenvalue, for all eigenvalues is diagonalizable; it can be written as $A = PDP^{-1}$ where D is a diagonal matrix:

$$D = \begin{bmatrix} * & 0 & \cdots & 0 \\ 0 & * & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & * \end{bmatrix}$$

In fact, the eigenspaces help produce the matrices P and D:

- **To find P, glue the vectors together from the bases of the eigenspaces.**
- **D will be the diagonal matrix whose entries are the eigenvalues of A, where the ith column of P belongs to the eigenvalue $D_{i,i}$.**

If the eigenspace of even one eigenvalue has dimension less than the eigenvalue's multiplicity, the matrix is not diagonalizable.

Example 1. **Let** $A = \begin{bmatrix} 3 & 1 \\ 4 & 3 \end{bmatrix}$**. The eigenvalues of** A **are 1 and 5, each with multiplicity 1. A basis for the eigenspace for** $\lambda = 1$ **is** $\left\{ \begin{bmatrix} -1/2 \\ 1 \end{bmatrix} \right\}$**, and a basis for the eigenspace for** $\lambda = 5$ **is** $\left\{ \begin{bmatrix} 1/2 \\ 1 \end{bmatrix} \right\}$**. Every eigenspace has the "right dimension", so the matrix** A **is diagonalizable.**

$$\lambda = 1\textbf{:}\ \left\{ \begin{bmatrix} -1/2 \\ 1 \end{bmatrix} \right\}\textbf{;} \qquad \lambda = 5\textbf{:}\ \left\{ \begin{bmatrix} 1/2 \\ 1 \end{bmatrix} \right\}$$

$$P = \begin{bmatrix} -1/2 & 1/2 \\ 1 & 1 \end{bmatrix} \textbf{ and } D = \begin{bmatrix} ? & ? \\ ? & ? \end{bmatrix}$$

$$\lambda = 1\textbf{:}\ \left\{ \begin{bmatrix} -1/2 \\ 1 \end{bmatrix} \right\}\textbf{;} \qquad \lambda = 5\textbf{:}\ \left\{ \begin{bmatrix} 1/2 \\ 1 \end{bmatrix} \right\}$$

$$P = \begin{bmatrix} -1/2 & 1/2 \\ 1 & 1 \end{bmatrix} \textbf{ and } D = \begin{bmatrix} ? & 0 \\ 0 & ? \end{bmatrix}$$

$$\lambda = 1\textbf{:}\ \left\{\begin{bmatrix} -1/2 \\ 1 \end{bmatrix}\right\}\textbf{;} \qquad \lambda = 5\textbf{:}\ \left\{\begin{bmatrix} 1/2 \\ 1 \end{bmatrix}\right\}$$

$$P = \begin{bmatrix} \mathbf{-1/2} & 1/2 \\ \mathbf{1} & 1 \end{bmatrix} \textbf{ and } D = \begin{bmatrix} \mathbf{?} & 0 \\ 0 & ? \end{bmatrix}$$

$$\lambda = \mathbf{1}\textbf{:}\ \left\{ \begin{bmatrix} -1/2 \\ 1 \end{bmatrix} \right\}\textbf{;} \qquad \lambda = 5\textbf{:}\ \left\{ \begin{bmatrix} 1/2 \\ 1 \end{bmatrix} \right\}$$

$$P = \begin{bmatrix} \mathbf{-1/2} & 1/2 \\ \mathbf{1} & 1 \end{bmatrix} \textbf{ and } D = \begin{bmatrix} \mathbf{1} & 0 \\ 0 & ? \end{bmatrix}$$

$$\lambda = 1\textbf{:}\ \left\{\begin{bmatrix} -1/2 \\ 1 \end{bmatrix}\right\}\textbf{;} \qquad \lambda = 5\textbf{:}\ \left\{\begin{bmatrix} 1/2 \\ 1 \end{bmatrix}\right\}$$

$$P = \begin{bmatrix} -1/2 & \mathbf{1/2} \\ 1 & \mathbf{1} \end{bmatrix} \textbf{ and } D = \begin{bmatrix} 1 & 0 \\ 0 & \textbf{?} \end{bmatrix}$$

$$\lambda = 1\textbf{:}\ \left\{\begin{bmatrix} -1/2 \\ 1 \end{bmatrix}\right\}\textbf{;} \qquad \lambda = \mathbf{5}\textbf{:}\ \left\{\begin{bmatrix} 1/2 \\ 1 \end{bmatrix}\right\}$$

$$P = \begin{bmatrix} -1/2 & \mathbf{1/2} \\ 1 & \mathbf{1} \end{bmatrix} \textbf{ and } D = \begin{bmatrix} 1 & 0 \\ 0 & \mathbf{5} \end{bmatrix}$$

$$\lambda = 1\textbf{:}\ \left\{ \begin{bmatrix} -1/2 \\ 1 \end{bmatrix} \right\}\textbf{;} \qquad \lambda = 5\textbf{:}\ \left\{ \begin{bmatrix} 1/2 \\ 1 \end{bmatrix} \right\}$$

$$P = \begin{bmatrix} -1/2 & 1/2 \\ 1 & 1 \end{bmatrix} \textbf{ and } D = \begin{bmatrix} 1 & 0 \\ 0 & 5 \end{bmatrix}$$

$$\textbf{(or } P = \begin{bmatrix} 1/2 & -1/2 \\ 1 & 1 \end{bmatrix} \textbf{ and } D = \begin{bmatrix} 5 & 0 \\ 0 & 1 \end{bmatrix}\textbf{.)}$$

Example 1 (continued). Let's check the first possibility, $P = \begin{bmatrix} -1/2 & 1/2 \\ 1 & 1 \end{bmatrix}$ **and** $D = \begin{bmatrix} 1 & 0 \\ 0 & 5 \end{bmatrix}$**:**

$$
\begin{aligned}
PDP^{-1} &= \begin{bmatrix} -1/2 & 1/2 \\ 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ 0 & 5 \end{bmatrix} \cdot \begin{bmatrix} -1/2 & 1/2 \\ 1 & 1 \end{bmatrix}^{-1} \\
&= \begin{bmatrix} -1/2 & 1/2 \\ 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ 0 & 5 \end{bmatrix} \cdot \begin{bmatrix} -1 & 1/2 \\ 1 & 1/2 \end{bmatrix} \\
&= \begin{bmatrix} -1/2 & 1/2 \\ 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} -1 & 1/2 \\ 5 & 5/2 \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ 4 & 3 \end{bmatrix} = A
\end{aligned}
$$

Example 2. The matrix $B = \begin{bmatrix} 2 & -1 & -3 & 9 \\ 0 & 2 & 0 & 9 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$ **is not diagonalizable. The eigenvalue 2 has multiplicity 2, but the eigenspace belonging to** $\lambda = 2$ **only has dimension 1.**

Example 3. Suppose a basis for the eigenspace of $\lambda = 2$ for C (a 3×3 matrix) is $\left\{ \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix} \right\}$, and a basis for the eigenspace of $\lambda = 3$ for C is $\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$.

Is C diagonalizable? If so, find P and D; otherwise, explain why not.

Example 3. Suppose a basis for the eigenspace of $\lambda = 2$ **for** C **(a** 3×3 **matrix) is** $\left\{ \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix} \right\}$**, and a basis for the eigenspace of** $\lambda = 3$ **for** C **is** $\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$**.**

Is C **diagonalizable? Yes.**

$$P = \begin{bmatrix} 2 & -3 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix} \text{ and } D = \begin{bmatrix} ? & ? & ? \\ ? & ? & ? \\ ? & ? & ? \end{bmatrix}$$

Example 3. $\lambda = 2$**:** $\left\{ \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix} \right\}$**;** $\lambda = 3$**:** $\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$

$P = \begin{bmatrix} 2 & -3 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$ **and** $D = \begin{bmatrix} \mathbf{?} & 0 & 0 \\ 0 & ? & 0 \\ 0 & 0 & ? \end{bmatrix}$

Example 3. $\lambda = 2$**:** $\left\{ \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix} \right\}$**;** $\lambda = 3$**:** $\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$

$P = \begin{bmatrix} 2 & -3 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$ **and** $D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & \textbf{?} & 0 \\ 0 & 0 & ? \end{bmatrix}$

Example 3. $\lambda = 2$**:** $\left\{ \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix} \right\}$**;** $\lambda = 3$**:** $\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$

$P = \begin{bmatrix} 2 & -3 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$ **and** $D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & \textbf{?} \end{bmatrix}$

Example 3. $\lambda = 2$: $\left\{ \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix} \right\}$; $\lambda = 3$: $\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$

$P = \begin{bmatrix} 2 & -3 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$ **and** $D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$

So why do we care about diagonalization?

- If $A = PDP^{-1}$, then

$$\begin{aligned} A^3 &= (PDP^{-1})(PDP^{-1})(PDP^{-1}) \\ &= PD(P^{-1}P)D(P^{-1}P)DP^{-1} = PDIDIDP^{-1} \\ &= PD^3P^{-1} \end{aligned}$$

  and $A^k = PD^kP^{-1}$ in general.

- $$\begin{bmatrix} a_1 & 0 & \cdots & 0 \\ 0 & a_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_n \end{bmatrix}^k = \begin{bmatrix} (a_1)^k & 0 & \cdots & 0 \\ 0 & (a_2)^k & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & (a_n)^k \end{bmatrix}$$

NOTE: This only works for diagonal matrices!!!

Example 1 (revisited). **Find** $\begin{bmatrix} 3 & 1 \\ 4 & 3 \end{bmatrix}^{100}$

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Solution: A **can be diagonalized using** $P = \begin{bmatrix} -1/2 & 1/2 \\ 1 & 1 \end{bmatrix}$ **and** $D = \begin{bmatrix} 1 & 0 \\ 0 & 5 \end{bmatrix}$**. Then**

$$\begin{aligned} A^{100} &= PD^{100}P^{-1} \\ &= \begin{bmatrix} -1/2 & 1/2 \\ 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1^{100} & 0 \\ 0 & 5^{100} \end{bmatrix} \cdot \begin{bmatrix} -1 & 1/2 \\ 1 & 1/2 \end{bmatrix} \\ &= \begin{bmatrix} -1/2 & 1/2 \\ 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} -1 & 1/2 \\ 5^{100} & 5^{100}/2 \end{bmatrix} \\ &= \begin{bmatrix} (1+5^{100})/2 & (5^{100}-1)/4 \\ \\ 5^{100}-1 & (1+5^{100})/2 \end{bmatrix} \end{aligned}$$

Example 1 (revisited). Find $\sqrt{\begin{bmatrix} 3 & 1 \\ 4 & 3 \end{bmatrix}}$**, a matrix** S **such that** $S^2 = A$**.**

Example 1 (revisited). Find $\sqrt{\begin{bmatrix} 3 & 1 \\ 4 & 3 \end{bmatrix}}$, a matrix S such that $S^2 = A$.

Solution: $A^k = PD^kP^{-1}$ even if k is not an integer. Here we have two choices for what $1^{1/2}$ would be: $+1$ or -1, and two choices for what $5^{1/2}$ would be. There are thus four possible answers for S!

If we say $1^{1/2} = -1$ and $5^{1/2} = \sqrt{5}$, then

$$S = P \cdot \begin{bmatrix} -1 & 0 \\ 0 & \sqrt{5} \end{bmatrix} \cdot P^{-1} = \begin{bmatrix} \dfrac{-1+\sqrt{5}}{2} & \dfrac{1+\sqrt{5}}{4} \\ 1+\sqrt{5} & \dfrac{-1+\sqrt{5}}{2} \end{bmatrix}$$

For Test 2, you should know how to:

- **determine whether a vector $\vec{u}$ is in the span of a set of vectors [4.2]**
- **determine whether a set of vectors is linearly independent, and if it is not:**
  - **find a linearly independent set of vectors whose span is the same as that of the original set [4.2] and**
  - **find a non-trivial linear combination of the vectors that adds up to $\vec{0}$ [4.2]**
- **find the dimension of a subspace, given a set of vectors whose span is that subspace [4.3]**

$\Longrightarrow$

For Test 2, you should know how to:

- **given a matrix A, find bases for the null space, the row space, and the column space of A, as well as the rank and nullity of A [4.4]**
- **given a basis and a vector, find the coordinates of the vector with respect to that basis [7.3]**
- **given the coordinates of a vector with respect to a basis, find the coordinates with respect to another basis [7.3]**
- **find the eigenvalues and eigenvectors of a matrix A [6.1]**
- **determine whether a matrix is diagonalizable [6.2]**

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