4.4. [Null Spaces and] Row and Column Spaces

Given a matrix A, what kind of a set is $N(A) = \left\{\vec{v} : A\vec{v} = \vec{0}\right\}$?

If we assume that $\vec{u}$ and $\vec{v}$ are in $N(A)$, then we know that $A\vec{u} = \vec{0}$ and $A\vec{v} = \vec{0}$; thus

$$A(\vec{u} + \vec{v}) = A\vec{u} + A\vec{v} = \vec{0} + \vec{0} = \vec{0},$$

which means $\vec{u} + \vec{v}$ is in $N(A)$. If r is a real number, then

$$A(r\vec{u}) = r(A\vec{u}) = r \cdot \vec{0} = \vec{0},$$

which means $r\vec{u}$ is in $N(A)$. Thus $N(A)$ is a subspace, called the null space of A.

How do we find a basis for the null space of the specific matrix $A = \begin{bmatrix} 1 & -2 & 3 & -1 & 4 \\ 1 & 1 & -1 & -2 & 3 \\ 1 & 7 & -9 & -4 & 1 \end{bmatrix}$**?**

How do we find a basis for the null space of the specific matrix $A = \begin{bmatrix} 1 & -2 & 3 & -1 & 4 \\ 1 & 1 & -1 & -2 & 3 \\ 1 & 7 & -9 & -4 & 1 \end{bmatrix}$**?**

Solve the equation $A\vec{x} = \vec{0}$**, or the system**

$$\begin{aligned} x_1 - 2x_2 + 3x_3 - x_4 + 4x_5 &= 0 \\ x_1 + x_2 - x_3 - 2x_4 + 3x_5 &= 0 \\ x_1 + 7x_2 - 9x_3 - 4x_4 + x_5 &= 0 \end{aligned}$$

The reduced row echelon form of the matrix

$$\left[\begin{array}{ccccc|c} 1 & -2 & 3 & -1 & 4 & 0 \\ 1 & 1 & -1 & -2 & 3 & 0 \\ 1 & 7 & -9 & -4 & 1 & 0 \end{array}\right]$$

is $$\left[\begin{array}{ccccc|c} 1 & 0 & 1/3 & -5/3 & 10/3 & 0 \\ 0 & 1 & -4/3 & -1/3 & -1/3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Equations for the variables are:

$$
\begin{aligned}
x_1 &= -\frac{1}{3}\,\alpha + \frac{5}{3}\,\beta - \frac{10}{3}\,\gamma \\
x_2 &= \frac{4}{3}\,\alpha + \frac{1}{3}\,\beta + \frac{1}{3}\,\gamma \\
x_3 &= \alpha \\
x_4 &= \beta \\
x_5 &= \gamma
\end{aligned}
$$

Making $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \alpha \cdot \begin{bmatrix} -1/3 \\ 4/3 \\ 1 \\ 0 \\ 0 \end{bmatrix} + \beta \cdot \begin{bmatrix} 5/3 \\ 1/3 \\ 0 \\ 1 \\ 0 \end{bmatrix} + \gamma \cdot \begin{bmatrix} -10/3 \\ 1/3 \\ 0 \\ 0 \\ 1 \end{bmatrix}$

Then $\left\{ \begin{bmatrix} -1/3 \\ 4/3 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 5/3 \\ 1/3 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -10/3 \\ 1/3 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$ **is a basis for** $N(A)$**.**

What is the dimension of $N(A)$ **here?**

Making $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \alpha \cdot \begin{bmatrix} -1/3 \\ 4/3 \\ 1 \\ 0 \\ 0 \end{bmatrix} + \beta \cdot \begin{bmatrix} 5/3 \\ 1/3 \\ 0 \\ 1 \\ 0 \end{bmatrix} + \gamma \cdot \begin{bmatrix} -10/3 \\ 1/3 \\ 0 \\ 0 \\ 1 \end{bmatrix}$

Then $\left\{ \begin{bmatrix} -1/3 \\ 4/3 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 5/3 \\ 1/3 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -10/3 \\ 1/3 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$ **is a basis for** $N(A)$**.**

What is the dimension of $N(A)$ **here? 3. The dimension of** $N(A)$ **is called the nullity of** A**.**

If $\vec{b}$ is not the zero vector, then the set of solutions to $A\vec{x} = \vec{b}$ is NOT a subspace. (It fails both parts of the Subspace Test, in fact.) However, the general solution does have a nice form, which you will meet (again?) in your Differential Equations class . . .

The solutions to $A\vec{x} = \vec{b}$**, where** $A = \begin{bmatrix} 1 & -2 & 3 & -1 & 4 \\ 1 & 1 & -1 & -2 & 3 \\ 1 & 7 & -9 & -4 & 1 \end{bmatrix}$ **and** $\vec{b} = \begin{bmatrix} 2 \\ -1 \\ -7 \end{bmatrix}$**, look like:**

$$\begin{bmatrix} 0 \\ -1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \alpha \cdot \begin{bmatrix} -1/3 \\ 4/3 \\ 1 \\ 0 \\ 0 \end{bmatrix} + \beta \cdot \begin{bmatrix} 5/3 \\ 1/3 \\ 0 \\ 1 \\ 0 \end{bmatrix} + \gamma \cdot \begin{bmatrix} -10/3 \\ 1/3 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

(This is the "expanded form" of the solution of a system of linear equations.)

$$\begin{bmatrix} 0 \\ -1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \boldsymbol{\alpha} \cdot \begin{bmatrix} -1/3 \\ 4/3 \\ 1 \\ 0 \\ 0 \end{bmatrix} + \boldsymbol{\beta} \cdot \begin{bmatrix} 5/3 \\ 1/3 \\ 0 \\ 1 \\ 0 \end{bmatrix} + \boldsymbol{\gamma} \cdot \begin{bmatrix} -10/3 \\ 1/3 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

This portion of the form is the general form of a vector $\vec{x}$ for which $A\vec{x} = \vec{0}$. In Differential Equations, you call this the "homogeneous solution."

$$\begin{bmatrix} \mathbf{0} \\ -\mathbf{1} \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix} + \alpha \cdot \begin{bmatrix} -1/3 \\ 4/3 \\ 1 \\ 0 \\ 0 \end{bmatrix} + \beta \cdot \begin{bmatrix} 5/3 \\ 1/3 \\ 0 \\ 1 \\ 0 \end{bmatrix} + \gamma \cdot \begin{bmatrix} -10/3 \\ 1/3 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

This portion of the form is an actual solution to $A\vec{x} = \vec{b}$; in fact, it's the solution you get by letting α, β, and γ all be zero. It is called a "particular solution."

Thus, if you have a "linear" equation, then you can write a general, arbitrary, solution in the form

$$\text{(particular solution)} + \text{(homogeneous solution)}$$

where the particular solution is an actual solution, and the homogenous solution is the solution to the equation when you replace something with zero. (This is how you obtain "homogeneous" equations in general.)

Okay, end of digression . . .

There are two other subspaces associated with a matrix A:

The **row space** of A is the span of the rows of A, and is denoted $R(A)$.

The **column space** of A is the span of the columns of A, and is denoted $C(A)$.

Since these sets are "spans of a set of vectors", they are automatically subspaces. So they each have a basis. So let's find a basis for each of these subspaces.

Row operations do not change the row space of a matrix. Also, Gaussian Elimination automatically converts "redundant" rows to all-zero rows. So a basis for the row space of A is just the non-zero rows in the reduced row echelon form of A.

The reduced row echelon form of

$$A = \begin{bmatrix} 1 & -2 & 3 & -1 & 4 \\ 1 & 1 & -1 & -2 & 3 \\ 1 & 7 & -9 & -4 & 1 \end{bmatrix} \textbf{ is } R = \begin{bmatrix} 1 & 0 & 1/3 & -5/3 & 10/3 \\ 0 & 1 & -4/3 & -1/3 & -1/3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

so a basis for the row space of A is

$$\{[1,\ 0,\ 1/3,\ -5/3,\ 10/3], [0,\ 1,\ -4/3,\ -1/3,\ -1/3]\}.$$

We actually already know how to find a basis for the column space of A; we use the same procedure as finding the basis for a subspace which is spanned by a set of (column) vectors. A basis for the column space of A will be the set of columns of A which have a pivot in their column of R.

$$A = \begin{bmatrix} \mathbf{1} & \mathbf{-2} & 3 & -1 & 4 \\ \mathbf{1} & \mathbf{1} & -1 & -2 & 3 \\ \mathbf{1} & \mathbf{7} & -9 & -4 & 1 \end{bmatrix} \qquad R = \begin{bmatrix} \mathbf{1} & 0 & 1/3 & -5/3 & 10/3 \\ 0 & \mathbf{1} & -4/3 & -1/3 & -1/3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

A basis for $C(A)$ is $\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ 7 \end{bmatrix} \right\}$.

What is the dimension of the row space of A?

What is the dimension of the column space of A?

What is the dimension of the row space of A? **2**

What is the dimension of the column space of A? **2**

Is this a coincidence?

For any matrix A, the dimension of the row space and the dimension of the column space are the same. This number is called the rank of A.

It equals the number of pivots in the reduced row echelon form of A.

It is the number of lead variables in the system of linear equations $A\vec{x} = \vec{b}$.

The nullity of A is the number of free variables in the system of linear equations $A\vec{x} = \vec{b}$. Thus:

$$\textbf{rank}\, A + \textbf{nullity}\, A = n,$$

where A is an $m \times n$ matrix.

The rank of the matrix A can be thought of as a measure of "how close A is to being invertible."

The rank of an $m \times n$ matrix will be between 0 and $\min(m, n)$. If a matrix has a rank of $\min(m, n)$, it is sometimes said to have full rank.

The only matrices with rank 0 are the zero matrices.

On the other end of the spectrum, the only $n \times n$ matrices which have rank n are those that have inverses.

$\square$