

4.2. Linear Combinations and Linear Independence

If we know that the vectors $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_k$ are in a subspace W , then the Subspace Test gives us more vectors which must also be in W ; for instance, $\vec{v}_1 + \vec{v}_2, 3\vec{v}_1, -2\vec{v}_2, \vec{v}_1 + \vec{v}_3$. But if we know that **THESE** vectors are in W , we can find even more:

$$(\vec{v}_1 + \vec{v}_2) + (3\vec{v}_1) = 4\vec{v}_1 + \vec{v}_2$$

and

$$(-2\vec{v}_2) + (\vec{v}_1 + \vec{v}_3) = \vec{v}_1 - 2\vec{v}_2 + \vec{v}_3,$$

for instance.

For instance, if $\vec{v}_1 = \begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix}$ and $\vec{v}_2 = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$ are in a subspace W , then we know that the vectors below are also in W :

$$\begin{bmatrix} 3 \\ -2 \\ 3 \end{bmatrix}, \begin{bmatrix} 6 \\ -9 \\ 3 \end{bmatrix}, \begin{bmatrix} -2 \\ -2 \\ -4 \end{bmatrix}, \begin{bmatrix} 9 \\ -11 \\ 6 \end{bmatrix}, \begin{bmatrix} 0 \\ -5 \\ -3 \end{bmatrix}$$

but the following vectors don't need to be in W :

$$\begin{bmatrix} -4 \\ -5 \\ 8 \end{bmatrix}, \begin{bmatrix} -11 \\ 3 \\ 3 \end{bmatrix}, \begin{bmatrix} 13 \\ 10 \\ -9 \end{bmatrix}, \begin{bmatrix} 9 \\ 0 \\ -4 \end{bmatrix}, \begin{bmatrix} -6 \\ -2 \\ 7 \end{bmatrix}$$

There doesn't seem to be any pattern to show which vectors must also be in W and which ones are not required to be in W . (Note that they might be in W anyway; W could be R^3 .)

If you look at the symbolic forms (\vec{v}_1) as opposed to the numerical form ($\begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix}$), a pattern **does** become apparent ...

Repeated calculations shows that any expression of the form

$$r_1\vec{v}_1 + r_2\vec{v}_2 + \cdots + r_k\vec{v}_k$$

will be in W , if the vectors $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_k$ are known to be in W . This type of expression shows up repeatedly when looking at vector spaces, so it is given a name; it is a **linear combination** of the vectors $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_k$.

If $S = \{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_k\}$ is a set of vectors (not necessarily a subspace), then the set of all linear combinations of $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_k$ is called the **span** of S , denoted $\text{span}(S)$. Note that $\text{span}(S)$ is a subspace!

Question. If $S = \left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ -3 \\ 7 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \right\}$, then is

$\vec{u} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ in the span of S ?

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$\vec{u} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ in the span of S ?

This happens if we can find real numbers r_1 , r_2 , and r_3 such that

$$r_1 \cdot \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} + r_2 \cdot \begin{bmatrix} 2 \\ -3 \\ 7 \end{bmatrix} + r_3 \cdot \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

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$\vec{u} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ in the span of S ?

This happens if we can find real numbers r_1 , r_2 , and r_3 such that

$$\begin{bmatrix} r_1 + 2r_2 \\ 2r_1 - 3r_2 + r_3 \\ 7r_2 - r_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Question. If $S = \left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ -3 \\ 7 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \right\}$, then is

$\vec{u} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ in the span of S ?

This happens if the system of linear equations represented by the matrix

$$\left[\begin{array}{ccc|c} 1 & 2 & 0 & 1 \\ 2 & -3 & 1 & 1 \\ 0 & 7 & -1 & 1 \end{array} \right]$$

has at least one solution.

The reduced row echelon form of $\left[\begin{array}{ccc|c} 1 & 2 & 0 & 1 \\ 2 & -3 & 1 & 1 \\ 0 & 7 & -1 & 1 \end{array} \right]$ is
...

Time to run the TI-83 Emulator!!!

WARNING! WARNING! WARNING! WARNING!

The TI calculators have a bug in them: You cannot find the RREF of a matrix with more rows than columns!

To fool the calculator into finding the RREF, add enough columns of zeros to make the matrix square, and delete them after you are done:

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 0 \\ 3 & 4 & 0 \\ 5 & 6 & 0 \end{bmatrix} \xrightarrow[\text{RREF}]{\text{Calculator}} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

WARNING! WARNING! WARNING! WARNING!

The reduced row echelon form of $\left[\begin{array}{ccc|c} 1 & 2 & 0 & 1 \\ 2 & -3 & 1 & 1 \\ 0 & 7 & -1 & 1 \end{array} \right]$ is

$$\left[\begin{array}{ccc|c} 1 & 0 & 2/7 & 5/7 \\ 0 & 1 & -1/7 & 1/7 \\ 0 & 0 & 0 & 0 \end{array} \right].$$

The system has infinitely many solutions, but we only need one to have \vec{u} be in the span of S . If we wanted to find a solution, we choose a value for the parameter ($\alpha = 6$ gives us nice numbers) and do what we did back in Chapter 1. Then

$$r_1 = -1 \quad r_2 = 1 \quad r_3 = 6$$

gives us one linear combination that adds up to \vec{u} .

Bonus question: Which vectors are in $\text{span}(S)$ of

$$S = \left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ -3 \\ 7 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \right\}?$$

Bonus question: Which vectors are in $\text{span}(S)$ of

$$S = \left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ -3 \\ 7 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \right\}?$$

The vector $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$ is in $\text{span}(S)$ if the system
represented by the matrix

$$\left[\begin{array}{ccc|c} 1 & 2 & 0 & a \\ 2 & -3 & 1 & b \\ 0 & 7 & -1 & c \end{array} \right]$$

has at least one solution.

A row echelon form of $\left[\begin{array}{ccc|c} 1 & 2 & 0 & a \\ 2 & -3 & 1 & b \\ 0 & 7 & -1 & c \end{array} \right]$ is

$$\left[\begin{array}{ccc|c} 1 & 2 & 0 & a \\ 0 & 1 & -1/7 & -b/7 + 2a/7 \\ 0 & 0 & 0 & c + b - 2a \end{array} \right]$$

A row echelon form of $\left[\begin{array}{ccc|c} 1 & 2 & 0 & a \\ 2 & -3 & 1 & b \\ 0 & 7 & -1 & c \end{array} \right]$ is

$$\left[\begin{array}{ccc|c} 1 & 2 & 0 & a \\ 0 & 1 & -1/7 & -b/7 + 2a/7 \\ 0 & 0 & 0 & c + b - 2a \end{array} \right]$$

The system will have at least one solution as long

as $c + b - 2a = 0$; so the vector $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$ is in $\text{span}(S)$ if

$c + b = 2a$.

We've found a way to describe $\text{span}(S)$ (or a general subspace), as the set of all linear combinations of a finite set of vectors; however, this is not the most

efficient set of vectors. This is because, if $\vec{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$,

$\vec{v}_2 = \begin{bmatrix} 2 \\ -3 \\ 7 \end{bmatrix}$, and $\vec{v}_3 = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$, then

$$-2\vec{v}_1 + \vec{v}_2 + 7\vec{v}_3 = \vec{0}.$$

This equation is easy to check; I'll show how I got it later on.

$-2\vec{v}_1 + \vec{v}_2 + 7\vec{v}_3 = \vec{0}$ means $\vec{v}_2 = 2\vec{v}_1 - 7\vec{v}_3$, so that a generic linear combination of \vec{v}_1 , \vec{v}_2 , and \vec{v}_3

$$c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3$$

can also be written as

$$c_1\vec{v}_1 + c_2(2\vec{v}_1 - 7\vec{v}_3) + c_3\vec{v}_3 = (c_1 + 2c_2)\vec{v}_1 + (c_3 - 7c_2)\vec{v}_3,$$

which is a linear combination of just \vec{v}_1 and \vec{v}_3 ; the vector \vec{v}_2 is “redundant”. This type of redundancy is called **linear dependence**.

In general, a set of vectors $\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_k\}$ is **linearly dependent** if there are real numbers c_1, c_2, \dots, c_k , not all zero, such that

$$c_1\vec{v}_1 + c_2\vec{v}_2 + \dots + c_k\vec{v}_k = \vec{0}.$$

If the only linear combination of $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_k$ that adds up to the zero vector is the one where $c_1 = c_2 = \dots = c_k = 0$ (sometimes called the **trivial linear combination**), then the set $\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_k\}$ is **linearly independent**.

We can determine whether a set of vectors is linearly independent by setting up a system of linear equations and finding the number of solutions. If the system has infinitely many solutions, we can find a non-trivial linear combination that adds up to $\vec{0}$, and we can also determine which vectors should be removed to obtain a linearly independent set of vectors whose span is the same as the span of the original set of vectors.

Let's use the set of vectors $S = \{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ where

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 2 \\ -3 \\ 7 \end{bmatrix}, \text{ and } \vec{v}_3 = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}.$$

Solving $c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3 = \vec{0}$ can be done by solving the system of linear equations represented by the augmented matrix

$$\left[\begin{array}{ccc|c} 1 & 2 & 0 & 0 \\ 2 & -3 & 1 & 0 \\ 0 & 7 & -1 & 0 \end{array} \right].$$

The reduced row echelon form of this matrix is

$\left[\begin{array}{ccc|c} 1 & 0 & 2/7 & 0 \\ 0 & 1 & -1/7 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right]$. Since there is no pivot in the third column, the system of linear equations has infinitely many solutions, and S is linearly dependent.

The solutions to the system represented by

$$\left[\begin{array}{ccc|c} 1 & 0 & 2/7 & 0 \\ 0 & 1 & -1/7 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right] \text{ are}$$

$$c_1 = -\frac{2}{7} s$$

$$c_2 = \frac{1}{7} s$$

$$c_3 = s$$

If we choose $s = 7$, then $c_1 = -2$, $c_2 = 1$, and $c_3 = 7$; this gives us the non-trivial linear combination from before:

$$-2\vec{v}_1 + \vec{v}_2 + 7\vec{v}_3 = \vec{0}.$$

To get a set T of linearly independent vectors which has the same span as S , we can remove the vectors which do not have a pivot in their column. So $T = \{\vec{v}_1, \vec{v}_2\}$ is a linearly independent set of vectors such that $\text{span}(T) = \text{span}(S)$.

(Note that there are other sets which meet these conditions: $\{\vec{v}_1, \vec{v}_3\}$ and $\{\vec{v}_2, \vec{v}_3\}$ also happen to work.)

4.3. Bases for Vector Spaces*

A linearly independent set S of vectors that spans a subspace W is a special set: S generates the subspace, in the sense that every element of W can be written as a linear combination of elements of S . Also, S is efficient, in that no proper subset of S will do the same thing. As such, a linearly independent set that spans W gets a special name: it's a **basis** for W . Using the example from the previous section, $\{\vec{v}_1, \vec{v}_2\}$ is a basis for $\text{span}(\{\vec{v}_1, \vec{v}_2, \vec{v}_3\})$.

* The slides for 4.2 and 4.3 are short, so I've combined them.

A subspace can have many bases.* The subspace from the previous section has at least three: $\{\vec{v}_1, \vec{v}_2\}$, $\{\vec{v}_1, \vec{v}_3\}$, and $\{\vec{v}_2, \vec{v}_3\}$. It has even more, since $\{2\vec{v}_1, \vec{v}_2\}$, $\{-\vec{v}_2, 5\vec{v}_3\}$, and $\{\vec{v}_1 + \vec{v}_2, \vec{v}_2\}$ are also bases for the same subspace.

However, these bases all have one thing in common:

* This is the plural of “basis”.

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However, these bases all have one thing in common: They all have the same number of elements.

The number of elements in a basis for the subspace W is called the **dimension** of W . The subspace in our example has a dimension of 2.

* This is the plural of “basis”.

Dimension is a useful parameter to have, since it lets us figure out what all the subspaces of R^2 (the plane) are.

First, we start with the subspaces with dimension 0; for technical reasons, this is the subspace $\{\vec{0}\}$, even though there are no vectors to take a linear combination of.

Then we move on to the subspaces of dimension 1; a basis will look like $\{\vec{u}\}$, where \vec{u} is not the zero vector. The subspace includes all multiples of \vec{u} ; these are the only possible “linear combinations” of one vector. We get the line passing through the origin and \vec{u} . The subspaces of dimension 1 thus look like lines passing through the origin.

Lastly, we consider the subspaces of dimension 2, whose basis we'll call $\{\vec{u}, \vec{v}\}$. We get the line passing through the origin and \vec{u} , and the line passing through the origin and \vec{v} . However, we can also get to points by travelling in the \vec{u} direction for a while, then the \vec{v} direction. Since $\{\vec{u}, \vec{v}\}$ is linearly independent, these directions are not the same, and we can actually get to any point in the plane. So the only subspace of dimension 2 is R^2 .

There are no subspaces of R^2 with dimension 3 or higher. The reason for this is that any set with more than n vectors from R^n cannot be linearly independent (you would have to have two or more pivots in some row). So no basis made up of vectors from R^n can have more than n vectors in it, and so no subspace of R^n can have dimension greater than n .

So the only subspaces of R^2 are: R^2 itself, $\{\vec{0}\}$, and the lines passing through the origin. The only subspaces of R^3 are R^3 itself, $\{\vec{0}\}$, lines passing through the origin, and planes passing through the origin.

Exercise. Find a basis for the set W of all vectors of the form $\begin{bmatrix} a \\ 2a + b \\ -2a + 3b \\ a + 4c \end{bmatrix}$ and the dimension of W .

Notice that

$$\begin{bmatrix} a \\ 2a + b \\ -2a + 3b \\ a + 4c \end{bmatrix} = a \cdot \begin{bmatrix} 1 \\ 2 \\ -2 \\ 1 \end{bmatrix} + b \cdot \begin{bmatrix} 0 \\ 1 \\ 3 \\ 0 \end{bmatrix} + c \cdot \begin{bmatrix} 0 \\ 0 \\ 0 \\ 4 \end{bmatrix},$$

so $\left\{ \begin{bmatrix} 1 \\ 2 \\ -2 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 4 \end{bmatrix} \right\}$ is a spanning set of W .

Is it a basis though?

$\left\{ \begin{bmatrix} 1 \\ 2 \\ -2 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 4 \end{bmatrix} \right\}$ will be a basis for W if

$\left\{ \begin{bmatrix} 1 \\ 2 \\ -2 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 4 \end{bmatrix} \right\}$ is linearly independent.

The RREF of $\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ -2 & 3 & 0 & 0 \\ 1 & 0 & 4 & 0 \end{array} \right]$ is $\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right]$.

Thus $\left\{ \begin{bmatrix} 1 \\ 2 \\ -2 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 4 \end{bmatrix} \right\}$ is a basis for W .

What is the dimension of W ?

Thus $\left\{ \begin{bmatrix} 1 \\ 2 \\ -2 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 4 \end{bmatrix} \right\}$ is a basis for W .

What is the dimension of W ? **3.**

