1.5. Inverses of Matrices*

There are actually two things we need to do, to make the $X = \dfrac{B}{A}$ formula usable.

- **Find a matrix I such that $IM = M$ for all matrices M (where the product is defined);**
- **Given a matrix A, find a matrix C so that**

$$CA = I.$$

Then our idea will go through . . .

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* **And one last part of Section 1.4.**

$$
\begin{aligned}
AX &= B \\
C(AX) &= CB \\
(CA)X &= CB \\
IX &= CB \\
X &= CB
\end{aligned}
$$

So which matrix will make $IM = M$ for all matrices M? (In particular, what if $M = \begin{bmatrix} 1 & -1 & 2 \\ 2 & -2 & 8 \\ 1 & -2 & 0 \end{bmatrix}$?)

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$$\begin{array}{ccccc} I & \cdot & M & = & M \\ ? \times ? & & 3 \times 3 & & 3 \times 3 \end{array}$$

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$$\begin{array}{ccccc} I & \cdot & M & = & M \\ 3 \times 3 & & 3 \times 3 & & 3 \times 3 \end{array}$$

So I must be a 3×3 matrix. How should we fill in the entries?

So which matrix will make $IM = M$ for all matrices M? (In particular, what if $M = \begin{bmatrix} 1 & -1 & 2 \\ 2 & -2 & 8 \\ 1 & -2 & 0 \end{bmatrix}$?)

No, $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$ doesn't work $\ldots$

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & -1 & 2 \\ 2 & -2 & 8 \\ 1 & -2 & 0 \end{bmatrix} = \begin{bmatrix} 4 & -5 & 10 \\ 4 & -5 & 10 \\ 4 & -5 & 10 \end{bmatrix}$$

So which matrix will make $IM = M$ for all matrices M? (In particular, what if $M = \begin{bmatrix} 1 & -1 & 2 \\ 2 & -2 & 8 \\ 1 & -2 & 0 \end{bmatrix}$?)

However,

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & -1 & 2 \\ 2 & -2 & 8 \\ 1 & -2 & 0 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 2 \\ 2 & -2 & 8 \\ 1 & -2 & 0 \end{bmatrix}$$

(In fact, this is the only such matrix that works.)

In general, the $m \times m$ matrix

$$I_m = \begin{bmatrix} 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & \cdots & 0 & 1 \end{bmatrix}$$

has the property that $I_m A = A$, whenever A is $m \times n$. The matrix I_m is called the identity matrix.

This solves one of our problems.

Now, let's suppose we have a matrix A, and we want to find a matrix C such that $CA = I$. For technical reasons, we will also require that $AC = CA = I$; if we can find this matrix C is called the inverse of A and is denoted A^{-1}.

Note that the matrix A has to be square; that is, the number of rows and columns must be the same.

The 1×1 case is not interesting, so we'll move on to the 2×2 case.

Derivation For Illustrative Purposes Only

Suppose $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ **(where** a**,** b**,** c **and** d **are numbers), and** $C = \begin{bmatrix} x & y \\ z & w \end{bmatrix}$**. Then** $CA = I$ **means**

$$\begin{bmatrix} x & y \\ z & w \end{bmatrix} \cdot \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

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$$\begin{aligned} ax + cy &= 1 \\ bx + dy &= 0 \\ az + cw &= 0 \\ bz + dw &= 1 \end{aligned}$$

So we have to solve a system of linear equations.

Derivation For Illustrative Purposes Only

Solving this system yields the formulas

$$x = \frac{d}{ad-bc} \qquad y = -\frac{b}{ad-bc}$$
$$z = -\frac{c}{ad-bc} \qquad w = \frac{a}{ad-bc}$$

So $$\boxed{\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}}.$$

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What happens if $ad - bc = 0$? Then it turns out that the system of linear equations has no solutions, and A does not have an inverse.

More terminology: A matrix which has an inverse is called invertible or nonsingular. A matrix which does not have an inverse is called non-invertible or singular.

What happens with larger matrices? Since a system of linear equations arises in a natural way, you might suspect that we'll be doing Gauss-Jordan Elimination to find the inverse in general. You would be right ...

Procedure for Finding the Inverse of a General Matrix A

- **Set up the augmented matrix** $[\ A \mid I\ ]$**, where** I **is the identity matrix.**
- **Do Gauss-Jordan Elimination on this Matrix.**
- **If the reduced row echelon form looks like** $[\ I \mid C\ ]$**, then** $C = A^{-1}$**. Otherwise, the matrix** A **has no inverse.**

Example. Find the inverse of the matrix $\begin{bmatrix} 1 & -1 & 2 \\ 2 & -2 & 8 \\ 1 & -2 & 0 \end{bmatrix}$**,**

if it exists.

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We will do row reduction on the augmented matrix

$$\left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 2 & -2 & 8 & 0 & 1 & 0 \\ 1 & -2 & 0 & 0 & 0 & 1 \end{array}\right].$$

$$\left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 2 & -2 & 8 & 0 & 1 & 0 \\ 1 & -2 & 0 & 0 & 0 & 1 \end{array}\right]$$

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$$\xrightarrow[\textcircled{2} - 2\textcircled{1}]{} \left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 0 & 0 & 4 & -2 & 1 & 0 \\ 1 & -2 & 0 & 0 & 0 & 1 \end{array}\right]$$

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$$\xrightarrow[\substack{-\textcircled{2} \\ \frac{1}{4}\textcircled{3}}]{} \left[\begin{array}{rrr|rrr} 1 & -1 & 2 & 1 & 0 & 0 \\ 0 & 1 & 2 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1/2 & 1/4 & 0 \end{array}\right]$$

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$$\xrightarrow[\text{①} + \text{②}]{} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 4 & -1 & -1 \\ 0 & 1 & 0 & 2 & -1/2 & -1 \\ 0 & 0 & 1 & -1/2 & 1/4 & 0 \end{array}\right]$$

Since we have the identity matrix on the left-hand side of our final matrix

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 4 & -1 & -1 \\ 0 & 1 & 0 & 2 & -1/2 & -1 \\ 0 & 0 & 1 & -1/2 & 1/4 & 0 \end{array}\right]$$

, this means

$$\begin{bmatrix} 1 & -1 & 2 \\ 2 & -2 & 8 \\ 1 & -2 & 0 \end{bmatrix}^{-1} = \boxed{\begin{bmatrix} 4 & -1 & -1 \\ 2 & -1/2 & -1 \\ -1/2 & 1/4 & 0 \end{bmatrix}}.$$

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Okay; now that we have the inverse of our matrix, what can we do with it?

The system of linear equations

$$\begin{aligned} x - y + 2z &= 3 \\ 2x - 2y + 8z &= 22 \\ x - 2y &= 2 \end{aligned}$$

can be written in the form $AX = B$**, where**

$$A = \begin{bmatrix} 1 & -1 & 2 \\ 2 & -2 & 8 \\ 1 & -2 & 0 \end{bmatrix}, \quad X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \quad \textbf{and} \quad B = \begin{bmatrix} 3 \\ 22 \\ 2 \end{bmatrix}.$$

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and the solution to the system is

$$X = A^{-1}B = \begin{bmatrix} 4 & -1 & -1 \\ 2 & -1/2 & -1 \\ -1/2 & 1/4 & 0 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ 22 \\ 2 \end{bmatrix} = \boxed{\begin{bmatrix} -12 \\ -7 \\ 4 \end{bmatrix}}.$$

Some properties of the inverse of a matrix:

- $(A^{-1})^{-1} = A$
- **You CAN cancel invertible matrices: If** $AB = AC$**, and** A **is invertible, then** $B = C$**.**
- $(AB)^{-1} = B^{-1}A^{-1}$ **(not** $A^{-1}B^{-1}$**)**
- $(A^n)^{-1} = (A^{-1})^n$

And one more question: In general, do we have an indicator which will tell us whether a matrix is invertible (like $ad - bc$ **for** 2×2 **matrices)?**

□