1.4. Matrix Operations

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1.4. Matrix Arithmetic

Matrices also exist as objects in and of themselves, and it turns out we can "do things" to them, which have algebraic properties. This will enable us (eventually) to find another way to solve a system of linear equations.

But first, we give some definitions, so we know exactly what we're talking about . . .

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \\ 10 & 11 & 12 \end{bmatrix}$$

A matrix is a rectangular arrangment of numbers. (Parentheses ( ) can be used instead of brackets [ ] to group the entries of a matrix; however, vertical lines | | have a different meaning.)

$$A = \begin{bmatrix} 1 & 2 & 3 \\ \mathbf{4} & \mathbf{5} & \mathbf{6} \\ 7 & 8 & 9 \\ 10 & 11 & 12 \end{bmatrix}$$

A row consists of the entries in a horizontal "slice". (The second row of A is shown above in bold.)

$$A = \begin{bmatrix} 1 & 2 & \mathbf{3} \\ 4 & 5 & \mathbf{6} \\ 7 & 8 & \mathbf{9} \\ 10 & 11 & \mathbf{12} \end{bmatrix}$$

A column consists of the entries in a vertical "slice". The third column of A is shown above in bold.

A row vector is a matrix with only one row, such as

$$\begin{bmatrix} 1 & 2 & 3 & 4 \end{bmatrix}$$

A column vector (or just a vector) is a matrix with only one column, such as

$$\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \\ 10 & 11 & 12 \end{bmatrix}$$

The entry in the ith row and the j column of the matrix A is denoted $A_{i,j}$ (my notation) or $a_{i,j}$ (the book's notation). Thus $A_{3,1} = ?$.

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$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \\ 10 & 11 & 12 \end{bmatrix}$$

If the matrix A has m rows and n columns, it is said to be "an $m \times n$ matrix". These numbers ($m \times n$) are also the dimensions (or "size") of the matrix.

The dimensions of A (above) are ???

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If the matrix A has m rows and n columns, it is said to be "an $m \times n$ matrix". These numbers $(m \times n)$ are also the dimensions of the matrix.

The dimensions of A (above) are 4×3.

When are two matrices equal? (When can we write $A = B$?)

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When they have the same dimensions, and corresponding entries are the same. That is, when

$$A_{i,j} = B_{i,j}$$

for "reasonable" i and j.

Now, we move on to arithmetic of matrices. What do you think

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 0 & 1 \\ 0 & 1 \end{bmatrix} \textbf{ is?}$$

Now, we move on to arithmetic of matrices. What do you think

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 0 & 1 \\ 0 & 1 \end{bmatrix} \textbf{ is? } \begin{bmatrix} 1+0 & 2+1 \\ 3+0 & 4+1 \\ 5+0 & 6+1 \end{bmatrix} \textbf{ or } \begin{bmatrix} 1 & 3 \\ 3 & 5 \\ 5 & 7 \end{bmatrix}.$$

To be more precise, if A and B have the same dimensions, then the (i, j)th entry of $A+B$ is $A_{i,j}+B_{i,j}$. (If A and B have different dimensions, we don't bother defining $A + B$, and say that this expression is undefined.)

We can also multiply a matrix by a real number; this is called scalar multiplication.

$$-2\begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 2 \end{bmatrix} =$$

We can also multiply a matrix by a real number; this is called scalar multiplication.

$$-2\begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 2 \end{bmatrix} = \begin{bmatrix} -2 & -4 & 0 \\ 0 & -2 & -4 \end{bmatrix}.$$

To be more precise, if r **is a real number, and** A **is an** $m \times n$ **matrix, then** rA **is the** $m \times n$ **matrix whose** (i, j)**th entry is** $rA_{i,j}$**.**

These arithemtic operations have a lot of the expected properties.

$$
\begin{aligned}
B + A &= \\
A + (B + C) &= \\
r(A + B) &= \\
(r + s)A &= \\
r(sA) &= \\
1A &=
\end{aligned}
$$

These arithemtic operations have a lot of the expected properties.

$$
\begin{aligned}
B + A &= A + B \\
A + (B + C) &= (A + B) + C \\
r(A + B) &= (rA) + (rB) \\
(r + s)A &= (rA) + (sA) \\
r(sA) &= (rs)A \\
1A &= A
\end{aligned}
$$

In normal addition, we have a special number "0" with the property that $x + 0 = x$ for any x. Is there a similar object for matrix addition?

$$\begin{bmatrix} 1 & 2 \\ -1 & -1 \\ 3 & 5 \end{bmatrix} + M = \begin{bmatrix} 1 & 2 \\ -1 & -1 \\ 3 & 5 \end{bmatrix}$$

In normal addition, we have a special number "0" with the property that $x + 0 = x$ for any x. Is there a similar object for matrix addition?

$$\begin{bmatrix} 1 & 2 \\ -1 & -1 \\ 3 & 5 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ -1 & -1 \\ 3 & 5 \end{bmatrix}$$

This matrix is called the **zero matrix**, and I will denote it by $\mathbf{0}$. (A **bold** zero; a normally typset zero will be the number zero.) Note that there are actually infinitely many zero matrices, one of each dimension.

Note that $0A = \mathbf{0}$ for any matrix A.

Exercise: Find all ordered triples (x, y, z) of real numbers such that

$$z \cdot \begin{bmatrix} 0 & 0 & x \\ 0 & z & 0 \end{bmatrix} + \begin{bmatrix} 1 & x & y \\ 3 & 2 & x+y \end{bmatrix} = \begin{bmatrix} 1 & 0 & 2 \\ 3 & 6 & 2 \end{bmatrix}.$$

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Our strategy is to get an equation which looks like $A = B$; then we can compare entries of the matrices to find x, y, and z. We need to use matrix arithmetic to simplify the expression on the left-hand side.

$$z \cdot \begin{bmatrix} 0 & 0 & x \\ 0 & z & 0 \end{bmatrix} + \begin{bmatrix} 1 & x & y \\ 3 & 2 & x + y \end{bmatrix} = \begin{bmatrix} 1 & 0 & 2 \\ 3 & 6 & 2 \end{bmatrix}$$

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$$\begin{bmatrix} 0 & 0 & xz \\ 0 & z^2 & 0 \end{bmatrix} + \begin{bmatrix} 1 & x & y \\ 3 & 2 & x+y \end{bmatrix} = \begin{bmatrix} 1 & 0 & 2 \\ 3 & 6 & 2 \end{bmatrix}$$

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Comparing entries, we get the equations $1 = 1$**,** $x = 0$**,** $xz + y = 2$**,** $3 = 3$**,** $z^2 + 2 = 6$**, and** $x + y = 2$**.**

$$z \cdot \begin{bmatrix} 0 & 0 & x \\ 0 & z & 0 \end{bmatrix} + \begin{bmatrix} 1 & x & y \\ 3 & 2 & x+y \end{bmatrix} = \begin{bmatrix} 1 & 0 & 2 \\ 3 & 6 & 2 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 & xz \\ 0 & z^2 & 0 \end{bmatrix} + \begin{bmatrix} 1 & x & y \\ 3 & 2 & x+y \end{bmatrix} = \begin{bmatrix} 1 & 0 & 2 \\ 3 & 6 & 2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & x & xz+y \\ 3 & z^2+2 & x+y \end{bmatrix} = \begin{bmatrix} 1 & 0 & 2 \\ 3 & 6 & 2 \end{bmatrix}$$

Comparing entries, we get the equations $1 = 1$, $x = 0$, $xz + y = 2$, $3 = 3$, $z^2 + 2 = 6$, and $x + y = 2$. Since $x = 0$, we find out that $y = 2$; since $z^2 = 4$, $z = \pm 2$, so the solutions are $(x, y, z) = \boxed{(0, 2, 2)}$ and $\boxed{(0, 2, -2)}$.

Another example. This one is related to the system

$$\begin{array}{ccccc|c} x_1 & x_2 & x_3 & x_4 & x_5 & \\ \end{array}$$

$$\left[\begin{array}{rrrrr|r} 1 & 0 & -1 & 0 & 3 & 2 \\ 0 & 1 & -2 & 0 & 4 & 3 \\ 0 & 0 & 0 & 1 & -1 & -2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

We described the solutions of

$$\begin{array}{ccccc|c} x_1 & x_2 & x_3 & x_4 & x_5 & \\ \end{array}$$
$$\left[\begin{array}{ccccc|c} 1 & 0 & -1 & 0 & 3 & 2 \\ 0 & 1 & -2 & 0 & 4 & 3 \\ 0 & 0 & 0 & 1 & -1 & -2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

using these equations:

$$x_1 = 2 + \alpha - 3\beta$$
$$x_2 = 3 + 2\alpha - 4\beta$$
$$x_3 = \alpha$$
$$x_4 = -2 + \beta$$
$$x_5 = \beta$$

WeBWorK will not accept the solution in this form; we need to convert them into another form.

$$
\begin{aligned}
x_1 &= 2 + \alpha - 3\beta \\
x_2 &= 3 + 2\alpha - 4\beta \\
x_3 &= \alpha \\
x_4 &= -2 + \beta \\
x_5 &= \beta
\end{aligned}
$$

by writing the vector $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix}$ **in "expanded" form.**

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 2 + \alpha - 3\beta \\ 3 + 2\alpha - 4\beta \\ \alpha \\ -2 + \beta \\ \beta \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 2+\alpha-3\beta \\ 3+2\alpha-4\beta \\ \alpha \\ -2+\beta \\ \beta \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 0 \\ -2 \\ 0 \end{bmatrix} + \begin{bmatrix} \alpha \\ 2\alpha \\ \alpha \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} -3\beta \\ -4\beta \\ 0 \\ \beta \\ \beta \end{bmatrix}$$

This "decomposition" was done by writing the vector as the sum of three column vectors:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} \mathbf{2} + \alpha - 3\beta \\ \mathbf{3} + 2\alpha - 4\beta \\ \alpha \\ \mathbf{-2} + \beta \\ \beta \end{bmatrix} = \begin{bmatrix} \mathbf{2} \\ \mathbf{3} \\ \mathbf{0} \\ \mathbf{-2} \\ \mathbf{0} \end{bmatrix} + \begin{bmatrix} \alpha \\ 2\alpha \\ \alpha \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} -3\beta \\ -4\beta \\ 0 \\ \beta \\ \beta \end{bmatrix}$$

This "decomposition" was done by writing the vector as the sum of three column vectors: (1) the constant terms (using a 0 if there is no constant term),

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 2 + \boldsymbol{\alpha} - 3\beta \\ 3 + \mathbf{2}\boldsymbol{\alpha} - 4\beta \\ \boldsymbol{\alpha} \\ -2 + \beta \\ \beta \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 0 \\ -2 \\ 0 \end{bmatrix} + \begin{bmatrix} \boldsymbol{\alpha} \\ \mathbf{2}\boldsymbol{\alpha} \\ \boldsymbol{\alpha} \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix} + \begin{bmatrix} -3\beta \\ -4\beta \\ 0 \\ \beta \\ \beta \end{bmatrix}$$

This "decomposition" was done by writing the vector as the sum of three column vectors: (1) the constant terms, (2) the terms involving α**,**

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 2+\alpha \boldsymbol{-3\beta} \\ 3+2\alpha \boldsymbol{-4\beta} \\ \alpha \\ -2+\boldsymbol{\beta} \\ \boldsymbol{\beta} \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 0 \\ -2 \\ 0 \end{bmatrix} + \begin{bmatrix} \alpha \\ 2\alpha \\ \alpha \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} \boldsymbol{-3\beta} \\ \boldsymbol{-4\beta} \\ \mathbf{0} \\ \boldsymbol{\beta} \\ \boldsymbol{\beta} \end{bmatrix}$$

This "decomposition" was done by writing the vector as the sum of three column vectors. This "decomposition" was done by writing the vector as the sum of three column vectors: (1) the constant terms, (2) the terms involving α, and (3) the terms involving β.

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 2+\alpha-3\beta \\ 3+2\alpha-4\beta \\ \alpha \\ -2+\beta \\ \beta \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 0 \\ -2 \\ 0 \end{bmatrix} + \begin{bmatrix} \alpha \\ 2\alpha \\ \alpha \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} -3\beta \\ -4\beta \\ 0 \\ \beta \\ \beta \end{bmatrix}$$

Notice that: (1) These are the only terms which appear in the equations for the x_i's,

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 2+\alpha-3\beta \\ 3+2\alpha-4\beta \\ \alpha \\ -2+\beta \\ \beta \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 0 \\ -2 \\ 0 \end{bmatrix} + \begin{bmatrix} \alpha \\ 2\alpha \\ \alpha \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} -3\beta \\ -4\beta \\ 0 \\ \beta \\ \beta \end{bmatrix}$$

Notice that: (1) These are the only terms which appear in the equations for the x_i's, and (2) every term in the last two vectors is a multiple of α or a multiple of β (so we can factor it out).

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 0 \\ -2 \\ 0 \end{bmatrix} + \alpha \cdot \begin{bmatrix} 1 \\ 2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + \beta \cdot \begin{bmatrix} -3 \\ -4 \\ 0 \\ 1 \\ 1 \end{bmatrix}$$

This is the form of the solution that WeBWorK accepts.

Exercise: Find all solutions to the equation

$$2x - 3y + 4z = 6$$

and write the answer in expanded form.

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$$\left[\begin{array}{ccc|c} 2 & -3 & 4 & 6 \end{array}\right]$$

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and write the answer in expanded form.

$$\left[\begin{array}{ccc|c} 2 & -3 & 4 & 6 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & -\frac{3}{2} & 2 & 3 \end{array}\right]$$

The augmented matrix is now in reduced row echelon form!

$$\left[\begin{array}{ccc|c} 1 & -\frac{3}{2} & 2 & 3 \end{array}\right]$$

- **The free variables are** y **and** z**.**
- **Let** $y = s$ **and** $z = t$**.**
- $x = 3 + \frac{3}{2}\,y - 2z = 3 + \frac{3}{2}\,s - 2t$**.**

Then

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 + \dfrac{3}{2}s - 2t \\ s \\ t \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix} + s \cdot \begin{bmatrix} 3/2 \\ 1 \\ 0 \end{bmatrix} + t \cdot \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}$$

Now we move on to matrix multiplication. This is the "weird" arithmetic operation, and the question that usually comes up in students' minds is "Why would you bother defining matrix multiplication in this complicated way?" There are two good reasons; first, we haven't really used the "shape" of the matrix yet. Second, well, this is going to take us back to systems of linear equations. (So, until I get past the examples, please don't ask, "Why are we defining AB this way?" You'll learn everything you need to know, when you need to know it . . .)

First of all, when is the matrix product AB defined? Suppose that A is an $m \times n$ matrix and B is a $p \times q$ matrix.

Then AB is only defined if $n = p$ (when the number of columns of A equals the number of rows of B), and AB is then a $m \times q$ matrix.

Quick quiz: Which of the following matrix products defined, and what are the dimensions of the product, where A is 2×3, B is 3×2, C is 2×3, and D is 2×4?

- AB
- BA
- AC
- BD
- DB

Quick quiz: Which of the following matrix products defined, and what are the dimensions of the product, where A is 2×3, B is 3×2, C is 2×3, and D is 2×4?

- AB **Defined.** $2 \times (3 = 3) \times 2$. **Dimensions:** 2×2
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- AC
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- AB **Defined.** $2 \times (3 = 3) \times 2$. **Dimensions:** 2×2
- BA **Defined.** $3 \times (2 = 2) \times 3$. **Dimensions:** 3×3
- AC
- BD
- DB

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- BA **Defined.** $3 \times (2 = 2) \times 3$. **Dimensions:** 3×3
- AC **Undefined.** $2 \times (3 \neq 2) \times 3$.
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- AC **Undefined.** $2 \times (3 \neq 2) \times 3$.
- BD **Defined.** $3 \times (2 = 2) \times 4$. **Dimensions:** 3×4
- DB

Quick quiz: Which of the following matrix products defined, and what are the dimensions of the product, where A is 2×3, B is 3×2, C is 2×3, and D is 2×4?

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- AC **Undefined.** $2 \times (3 \neq 2) \times 3$.
- BD **Defined.** $3 \times (2 = 2) \times 4$. **Dimensions:** 3×4
- DB **Undefined.** $2 \times (4 \neq 3) \times 2$.

Next, we need to determine what the entries of AB are. To do this, we use a generalized version of the dot product, which tells us how to multiply a row vector by a column vector:

$$\begin{bmatrix} x_1 & x_2 & \cdots & x_k \end{bmatrix} \cdot \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_k \end{bmatrix} = x_1 y_1 + x_2 y_2 + \cdots + x_k y_k.$$

Now, I can give you the rule for finding one particular entry of AB: The (i, j)th entry (the one in the ith row and jth column of AB is the dot product of the ith row of A and the jth column of B.

(Fortunately, it's not backwards!)

Now it's time to throw in some numbers $\ldots$

$$\begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix} \cdot \begin{bmatrix} 3 & 3 \\ 1 & 2 \\ 0 & 1 \end{bmatrix} = ?$$

What are the dimensions of the final answer?

$$\begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix} \cdot \begin{bmatrix} 3 & 3 \\ 1 & 2 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} ? & ? \\ ? & ? \end{bmatrix}$$

What are the dimensions of the final answer? **2 × 2**

Now we find the entries of the matrix ...

$$\begin{bmatrix} \boxed{1 \quad 2 \quad 1} \\ 0 \quad 1 \quad 2 \end{bmatrix} \cdot \begin{bmatrix} \boxed{\begin{matrix} 3 \\ 1 \\ 0 \end{matrix}} & \begin{matrix} 3 \\ 2 \\ 1 \end{matrix} \end{bmatrix} = \begin{bmatrix} \boxed{?} & ? \\ ? & ? \end{bmatrix}$$

$$? = \begin{bmatrix} 1 & 2 & 1 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix} = 1 \cdot 3 + 2 \cdot 1 + 1 \cdot 0 = \mathbf{5}$$

$$\begin{bmatrix} \boxed{1 \quad 2 \quad 1} \\ 0 \quad 1 \quad 2 \end{bmatrix} \cdot \begin{bmatrix} 3 & \boxed{3} \\ 1 & \boxed{2} \\ 0 & \boxed{1} \end{bmatrix} = \begin{bmatrix} 5 & \boxed{?} \\ ? & ? \end{bmatrix}$$

$$? = \begin{bmatrix} 1 & 2 & 1 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} = 1 \cdot 3 + 2 \cdot 2 + 1 \cdot 1 = \mathbf{8}$$

$$\begin{bmatrix} 1 & 2 & 1 \\ \boxed{0 \quad 1 \quad 2} \end{bmatrix} \cdot \begin{bmatrix} 3 & 3 \\ 1 & 2 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 5 & 8 \\ \boxed{?} & ? \end{bmatrix}$$

$$? = \begin{bmatrix} 0 & 1 & 2 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix} = 0 \cdot 3 + 1 \cdot 1 + 2 \cdot 0 = \mathbf{1}$$

$$\begin{bmatrix} 1 & 2 & 1 \\ \boxed{0 \quad 1 \quad 2} \end{bmatrix} \cdot \begin{bmatrix} 3 & \boxed{3} \\ 1 & 2 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 5 & 8 \\ 1 & \boxed{?} \end{bmatrix}$$

$$? = [0 \ 1 \ 2] \cdot \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} = 0 \cdot 3 + 1 \cdot 2 + 2 \cdot 1 = \mathbf{4}$$

$$\begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix} \cdot \begin{bmatrix} 3 & 3 \\ 1 & 2 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 5 & 8 \\ 1 & 4 \end{bmatrix}$$

(Note that $\begin{bmatrix} 3 & 3 \\ 1 & 2 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix}$ **is the** 3×3 **matrix** $\begin{bmatrix} 3 & 9 & 9 \\ 1 & 4 & 5 \\ 0 & 1 & 2 \end{bmatrix}$**.)**

Matrix multiplication has the following "features":

- AB **is generally different from** BA**. In fact, any of the following can happen:**
  - AB **is defined and** BA **is undefined;**
  - AB **and** BA **are both defined but have different dimensions;**
  - AB **and** BA **are both defined, have the same dimensions, but the entries don't agree.**

More features of matrix multiplication:

- **If** $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ **and** $B = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$, **then** AB **and** BA **have the same dimensions, but the entries are different.**

- **Generally, if** $AC = BC$, **then you cannot conclude that** $A = B$. **In fact, you can even have** $A^2 = \mathbf{0}$ **with** $A \neq \mathbf{0}$; **for instance** $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$. ($A^2 = AA$.)

Matrix multiplication does have the following properties:

- $A(BC) = (AB)C$ **(provided that all multiplications are defined; and if one side is defined, the other is also defined)**
- $A(B + C) = (AB) + (AC)$ **(ditto)**
- $(A + B)C = (AC) + (BC)$ **(ditto)**
- $r(AB) = (rA)B = A(rB)$ **(ditto)**
- $A\mathbf{0} = \mathbf{0}$ **and** $\mathbf{0}A = \mathbf{0}$ **(ditto)**

Now . . . Why do we define matrix multiplication in this bizarre way?

Let's look at the matrix equation

$$\begin{bmatrix} 1 & -1 & 2 \\ 2 & -2 & 8 \\ 1 & -2 & 0 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 22 \\ 2 \end{bmatrix}$$

Let's multiply the left-hand side out.

$$\begin{bmatrix} x - y + 2z \\ 2x - 2y + 8z \\ x - 2y \end{bmatrix} = \begin{bmatrix} 3 \\ 22 \\ 2 \end{bmatrix}$$

This is another way of writing

$$\begin{aligned} x - y + 2z &= 3 \\ 2x - 2y + 8z &= 22 \\ x - 2y &= 2 \end{aligned}$$

which is just a system of linear equations!

We can reverse the process as well; **any system of linear equations can be written as a matrix equation**

$$AX = B.$$

The matrix A has the coefficients of the variables, X is the matrix of the variables, and B consists of the numbers on the right-hand side.

Now, **how do you solve the equation** $AX = B$ **for** X**?**

$$AX = B \quad \Longrightarrow \quad X = \text{``}\frac{B}{A}\text{''}$$

(Not exactly; there are a few things we need to clarify, before we get a working formula.)

□