1.2. Matrices and Gaussian Elimination

Suppose you need to solve the following system of linear equations:

$$\begin{aligned} x &- y + 2z = 3 \\ 2x &- 2y + 8z = 22 \\ x &- 2y = 2 \end{aligned}$$

We're going to be doing elimination over and over, which means writing equations over and over, which is a real pain, since only the numbers will be changing.

So we are going to convert the system into a matrix and work with the matrix instead:

$$\begin{aligned} x - y + 2z &= 3 \\ 2x - 2y + 8z &= 22 \\ x - 2y &= 2 \end{aligned} \quad \Longrightarrow \quad \left[\begin{array}{ccc|c} 1 & -1 & 2 & 3 \\ 2 & -2 & 8 & 22 \\ 1 & -2 & 0 & 2 \end{array}\right]$$

(A matrix in general is a rectangular arrangement of numbers.) Note that if a variable is missing from an equation, we put a 0 in the matrix. Also, I've drawn a vertical bar separating the coefficients of the variables from the numbers on the right-hand side. The right-most column is treated differently, and this is a reminder of that.

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Because it represents the system

$$\begin{aligned} x \qquad\qquad &= A \\ y \qquad &= B \\ z &= C \end{aligned}$$

which says that the solution is (A, B, C)**.**

Now, how do we get from

$$\left[\begin{array}{ccc|c} 1 & -1 & 2 & 3 \\ 2 & -2 & 8 & 22 \\ 1 & -2 & 0 & 2 \end{array}\right] \quad \textbf{to} \quad \left[\begin{array}{ccc|c} 1 & 0 & 0 & A \\ 0 & 1 & 0 & B \\ 0 & 0 & 1 & C \end{array}\right]$$

without changing the solution?

There are three types of "operations" (things we do to a matrix) which will not change the solutions when we do them, and these three types will ALWAYS allow us to put our matrix in the desired form!

1. Swap two rows of the matrix. For instance, if I swap the second and third rows of the matrix, I do the following:

$$\left[\begin{array}{ccc|c} 1 & -1 & 2 & 3 \\ 2 & -2 & 8 & 22 \\ 1 & -2 & 0 & 2 \end{array}\right] \Longrightarrow \left[\begin{array}{ccc|c} 1 & -1 & 2 & 3 \\ 1 & -2 & 0 & 2 \\ 2 & -2 & 8 & 22 \end{array}\right]$$

The book's notation for this row operation is $SWAP(R_2, R_3)$**.**

My notation for this row operation is $\textcircled{2} \leftrightarrow \textcircled{3}$**. (**$\textcircled{i}$ **represents the** i**th row in general.)**

2. Multiply a row by a nonzero number. For instance, if multiply the first row by 3, I do the following:

$$\left[\begin{array}{ccc|c} 1 & -1 & 2 & 3 \\ 2 & -2 & 8 & 22 \\ 1 & -2 & 0 & 2 \end{array}\right] \Longrightarrow \left[\begin{array}{ccc|c} 3 & -3 & 6 & 9 \\ 2 & -2 & 8 & 22 \\ 1 & -2 & 0 & 2 \end{array}\right]$$

The book's notation for this row operation is $R_1 = 3r_1$**.**

My notation for this row operation is $3①$**.**

3. Multiply a row by a (nonzero) number, and add this "virtual" row to another row. For instance, if multiply row 2 by -2, then add it to row 1, I do the following:

$$\left[\begin{array}{ccc|c} 1 & -1 & 2 & 3 \\ 2 & -2 & 8 & 22 \\ 1 & -2 & 0 & 2 \end{array}\right]$$

Row 1: $[\quad 1 \quad -1 \quad 2 \mid \quad 3]$

Row 2: $[\quad 2 \quad -2 \quad 8 \mid \quad 22]$

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Row 1:	$\left[\begin{array}{rrr|r} 1 & -1 & 2 & 3 \end{array}\right]$
Row 2:	$\left[\begin{array}{rrr|r} 2 & -2 & 8 & 22 \end{array}\right]$
$(-2)\times$ **Row 2:**	$\left[\begin{array}{rrr|r} -4 & 4 & -16 & -44 \end{array}\right]$

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Row 1:	1	-1	2	$\vert$	3
Row 2:	2	-2	8	$\vert$	22
$(-2)\times$ **Row 2:**	-4	4	-16	$\vert$	-44
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So row 1 gets replaced by $[\,-3 \quad 3 \quad -14 \mid -41\,]$**.**

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$$\left[\begin{array}{ccc|c} 1 & -1 & 2 & 3 \\ 2 & -2 & 8 & 22 \\ 1 & -2 & 0 & 2 \end{array}\right] \Longrightarrow \left[\begin{array}{rrr|r} -3 & 3 & -14 & -41 \\ 2 & -2 & 8 & 22 \\ 1 & -2 & 0 & 2 \end{array}\right]$$

The book's notation for this row operation is $R_1 = r_1 - 2R_2$.

My notation for this row operation is $① - 2②$.

So, to summarize, you can:

- Swap two rows
- Multiply (or divide) a row by a nonzero number
- Add a multiple of one row to another row

and get a new matrix whose system has the **same solutions** as the old one.

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This tells you the “rules of the game”, it gives you no idea on how to “win the game”, since later row operations can mess up work you’ve already done. So what is the strategy?

The first part of the strategy is to concentrate on the first column:

$$\left[\begin{array}{ccc|c} \mathbf{1} & -1 & 2 & 3 \\ \mathbf{2} & -2 & 8 & 22 \\ \mathbf{1} & -2 & 0 & 2 \end{array}\right]$$

and make the first column have a 1 at the top, and 0's underneath that 1. The complete matrix will then look like:

$$\left[\begin{array}{ccc|c} \mathbf{1} & * & * & * \\ \mathbf{0} & * & * & * \\ \mathbf{0} & * & * & * \end{array}\right]$$

and we'll be on our way.

$$\left[\begin{array}{ccc|c} \mathbf{1} & -1 & 2 & 3 \\ 2 & -2 & 8 & 22 \\ 1 & -2 & 0 & 2 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & -1 & 2 & 3 \\ \mathbf{2} & -2 & 8 & 22 \\ 1 & -2 & 0 & 2 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & -1 & 2 & 3 \\ 2 & -2 & 8 & 22 \\ 1 & -2 & 0 & 2 \end{array}\right] \xrightarrow[\substack{②-2① \\ R_2 = r_2 - 2R_1}]{} \left[\begin{array}{ccc|c} 1 & -1 & 2 & 3 \\ 0 & 0 & 4 & 16 \\ \mathbf{1} & -2 & 0 & 2 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & -1 & 2 & 3 \\ 2 & -2 & 8 & 22 \\ 1 & -2 & 0 & 2 \end{array}\right] \xrightarrow[R_2 = r_2 - 2R_1]{②-2①} \left[\begin{array}{ccc|c} 1 & -1 & 2 & 3 \\ 0 & 0 & 4 & 16 \\ 1 & -2 & 0 & 2 \end{array}\right]$$

$$\xrightarrow[R_3 = r_3 - R_1]{③-①} \left[\begin{array}{ccc|c} 1 & -1 & 2 & 3 \\ 0 & 0 & 4 & 16 \\ 0 & -1 & -2 & -1 \end{array}\right]$$

Now I have a matrix whose first column is in the form I want. Now for the second part of the strategy: Ignore the first row and first column of the matrix. In particular, do not use the first row any more! (Can you guess why you wouldn't want to?)

$$\left[\begin{array}{rrr|r} 1 & -1 & 2 & 3 \\ 0 & 0 & 4 & 16 \\ 0 & -1 & -2 & -1 \end{array}\right]$$

Now I have a matrix whose first column is in the form I want. Now for the second part of the strategy: Ignore the first row and first column of the matrix. In particular, **do not use the first row any more!** (Can you guess why you wouldn't want to?)

$$\left[\begin{array}{ccc|c} 1 & -1 & 2 & 3 \\ 0 & \mathbf{0} & 4 & 16 \\ 0 & \mathbf{-1} & -2 & -1 \end{array}\right]$$

Now for the third part of the strategy: If you have any entries left, go back to the first part. So now we have to fix up the 0 and the –1 in the matrix above. (The –1 in the first row, second column, is okay and doesn't have to be changed ... but I'll finish that thought in a few days.)

$$\left[\begin{array}{ccc|c} 1 & -1 & 2 & 3 \\ 0 & \mathbf{0} & 4 & 16 \\ 0 & -1 & -2 & -1 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & -1 & 2 & 3 \\ 0 & 0 & 4 & 16 \\ 0 & -1 & -2 & -1 \end{array}\right] \xrightarrow[\substack{②\leftrightarrow③ \\ SWAP(R_2,\,R_3)}]{} \left[\begin{array}{ccc|c} 1 & -1 & 2 & 3 \\ 0 & \mathbf{-1} & -2 & -1 \\ 0 & 0 & 4 & 16 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & -1 & 2 & 3 \\ 0 & 0 & 4 & 16 \\ 0 & -1 & -2 & -1 \end{array}\right]$$

$$\xrightarrow[\textcircled{2} \leftrightarrow \textcircled{3} \\ SWAP(R_2, R_3)]{} \left[\begin{array}{ccc|c} 1 & -1 & 2 & 3 \\ 0 & -1 & -2 & -1 \\ 0 & 0 & 4 & 16 \end{array}\right]$$

$$\xrightarrow[-\textcircled{2} \\ R_2 = -r_2]{} \left[\begin{array}{ccc|c} 1 & -1 & 2 & 3 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 4 & 16 \end{array}\right]$$

Then we ignore the first row and first column of the smaller matrix:

$$\left[\begin{array}{ccc|c} 1 & -1 & 2 & 3 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 4 & 16 \end{array}\right] \Longrightarrow \left[\begin{array}{ccc|c} 1 & -1 & 2 & 3 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 4 & 16 \end{array}\right]$$

Then we ignore the first row and first column of the smaller matrix:

$$\left[\begin{array}{ccc|c} 1 & -1 & 2 & 3 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 4 & 16 \end{array}\right] \implies \left[\begin{array}{ccc|c} 1 & -1 & 2 & 3 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 4 & 16 \end{array}\right]$$

and then fix up the 4. (If you're just solving a system of linear equations, you can stop here. However, there are other applications where you do want to have a 1 in place of this 4. So I'll keep going.)

$$\left[\begin{array}{ccc|c} 1 & -1 & 2 & 3 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & \mathbf{4} & 16 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & -1 & 2 & 3 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 4 & 16 \end{array}\right] \xrightarrow[R_3 = \frac{1}{4} r_3]{\frac{1}{4} \text{③}} \left[\begin{array}{ccc|c} 1 & -1 & 2 & 3 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & 4 \end{array}\right]$$

This “strategy” is called **Gaussian Elimination**, and the matrix we end up with in the end is in **row echelon form** (or **echelon form**, as the book calls it.)

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Now, let’s solve this system. We start with the bottom row and move upwards, using back substitution.

$$\left[\begin{array}{ccc|c} 1 & -1 & 2 & 3 \\ 0 & 1 & 2 & 1 \\ \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{4} \end{array}\right]$$

Row 3 states that $0 \cdot x + 0 \cdot y + 1 \cdot z = 4$**, so** $z = 4$**.**

$$\left[\begin{array}{ccc|c} 1 & -1 & 2 & 3 \\ \mathbf{0} & \mathbf{1} & \mathbf{2} & \mathbf{1} \\ 0 & 0 & 1 & 4 \end{array}\right]$$

Row 3 states that $0 \cdot x + 0 \cdot y + 1 \cdot z = 4$**, so** $z = 4$**. Row 2 states that**

$$y + 2z = 1,$$

so $y = 1 - 2z = 1 - 2 \cdot \mathbf{4} = -7$**.**

$$\left[\begin{array}{rrr|r} \mathbf{1} & \mathbf{-1} & \mathbf{2} & \mathbf{3} \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & 4 \end{array}\right]$$

Row 1 states that

$$x - y + 2z = 3,$$

so

$$x = 3 + y - 2z = 3 + (\mathbf{-7}) - 2 \cdot \mathbf{4} = -12,$$

making the solution $(x, y, z) = \boxed{(-12, -7, 4)}$**.**

$\square$