

An Investigation of Covariational Reasoning and Its Role in Learning the Concepts of Limit and Accumulation

Marilyn P. Carlson
Arizona State University
marilyn.carlson@asu.edu

Sean Larsen
Arizona State University
larsen@math.la.asu.edu

Sally Jacobs
Arizona State University
sally.jacobs@mcmail.maricopa.edu

Abstract: This study investigated the effect of research-based curricular materials on the development of first semester calculus students' covariational reasoning abilities. It also explored the role of covariational reasoning in the development of the concepts of limit and accumulation. We found evidence that the covariation curriculum was effective and that the reasoning abilities promoted in the activities were useful for completing select limit and accumulation tasks.

Introduction

Concepts of rate of change and changing rate of change are central to the study of calculus, but it is not clear how students acquire the ability to interpret and represent continuously changing rate for a function's domain. Further, little is known about the importance of covariational reasoning (coordinating images of two varying quantities and attending to the ways in which they change in relation to each other) in students' study of other concepts of calculus (e.g., limit and accumulation). This paper reports results from investigating calculus students' understanding of covariation. It extends the first author's (Carlson, 1998) and others' (Saldhana & Thompson, 1998) study of covariation and builds on previous research investigating the concepts of limit (e.g., Sierpiska, 1987a; Cornu, 1991; Tall, 1992; Cottrill et al., 1996; Szydlik, 2000) and accumulation (e.g., Thompson, 1994). Covariational reasoning has been shown to be an important ability for interpreting, describing and representing the behavior of a dynamic function event (Saldhana & Thompson, 1998; Carlson, 1998). In addition, Thompson (1994) has observed that images of rate are fundamental for reasoning about accumulation. However, no previous research has investigated the role of covariation in the development of students' understanding of the major conceptual strands of calculus. This paper reports on the covariational reasoning abilities of first semester calculus students and provides initial insights regarding the role of covariational reasoning in responding to specific limit and accumulation tasks.

Theoretical Model

The author's past study of function and covariation (Carlson, 1998) produced a framework for describing students' covariational reasoning abilities. This framework includes five categories of mental actions (Table 1) that have been observed in students when applying covariational reasoning in the context of representing and interpreting a graphical model of a dynamic function event. The initial image described in the framework is one of two variables changing simultaneously. This loose association undergoes multiple refinements as the student moves toward an image of increasing and decreasing rate over the entire domain of the function (Table 1). Describing these actions in the form of a framework provides a powerful tool with which to analyze covariational thinking to a finer degree than has been done in the past. It also provides structure and research based information for building curricular activities, as are described later in this paper.

Table 1. Covariation Framework

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| <p><i>MA1</i>) An image of two variables changing simultaneously;</p> <p><i>MA2</i>) A loosely coordinated image of how the variables are changing with respect to each other (e.g., increasing, decreasing);</p> <p><i>MA3</i>) An image of an amount of change of one variable while considering changes in discrete amounts of the other variable;</p> <p><i>MA4</i>) An image of rate/slope for contiguous intervals of the function's domain;</p> <p><i>MA5</i>) An image of continuously changing rate over the entire domain</p> |
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Methods

The subjects in this study were 24 first-semester calculus students at a large public university in the United States. The classroom was student-centered involving group-work and interactive lectures. The instructor for the course was the first author, with the second author observing students' thinking and noting

curricular effectiveness. A traditional text was supplemented with a set of curricular modules developed (by the authors) for each of the course's major conceptual strands. The development of each curricular module was guided by insights gained from the research literature on function, covariation, limit and accumulation. Each module included a collection of activities (both in-class and take-home) designed to engage students in sense making while building their understandings of these concepts. The data collection consisted of pre- and post- test assessments of students' covariational reasoning abilities. Follow-up interviews, designed to assess students' covariational reasoning and concept development, were conducted with 8 students after completing each curricular module and at the end of the semester. Interview subjects were selected to represent the diversity of the class' covariational reasoning abilities. Each interview subject had also been videotaped while completing the in-class activities in their individual groups.

The data analysis involved a comparison of pre- and post-test results. Interview and video-taped data were coded, categorized and classified using the covariation framework and general trends were observed and noted.

A Description of the Covariation Module

This module consisted of five separate activities designed to promote students' ability to attend to the covariant nature of dynamic functional relationships. Each activity contained a collection of prompts that encouraged students to coordinate an image of the two variables changing and to attend to and represent the way in which the independent and dependent variables changed in relationship to one another. Since previous results (Carlson, 1998) have reported that a process view of function is needed for applying covariational reasoning, each activity also included prompts that encouraged students to imagine a set of input values being acted upon by a process (or function-machine) to produce a set of output values. The first of the four activities prompted students to construct a maximal volume open-top box from an 8.5 x 11 piece of paper by cutting equal squares from the corners and folding up the sides. After construction of an algebraic model for this situation, students were prompted to determine both the box size for various values of x , and values of x for various box sizes. Also included in the set of writing exercises for this activity were prompts that focussed students' attention on the varying nature of the input and output variables. Students were asked to determine how changing the size of the square affected the size of the volume, and to specifically determine the range of values of x (the length of the side of the square) for which the box's volume was increasing. Written responses were required of each student, with class discussions emphasizing representational connections, in particular the connection between various boxes and corresponding points/values on the graph.

The second covariation activity presented a variety of distance-time graphs, followed by prompts for students to model, through their own motion, the behavior represented by these graphs and to produce a written description of their motion. With the use of motion detectors and a real-time display of the distance-time graph, students were able to acquire immediate feedback of the effect of their motion, and were encouraged to make adjustments as needed until their motion produced the desired graph. (Even though the terms concave up, concave down and inflection point had not been introduced, students were given graphs that changed from concave up to concave down and were expected to draw on their informal intuitions about speed and acceleration to determine how to walk these graphs).

The third activity took the form of a model-eliciting activity (Lesh, Hoover, Hole, Kelly, & Post, 2000), with the six principles of model-eliciting activities guiding its development. The activity was based in a realistic situation that required the construction of a mathematical model, while also requiring students to generalize and document their thinking. Students were also asked to produce a strategy guide for a physics student preparing for a lab exam in a physics class. They were instructed to produce a detailed set of general instructions to explain how to produce a "walking motion" for any possible graph.

The fourth activity was based on the familiar sliding ladder problem that asks students to determine the relative speed of the top of a 10-foot ladder as the base of the ladder is pulled away from the wall. Since these activities were completed early in the semester, the students did not have access to the tools of calculus, and therefore relied on algebraic and covariational reasoning when thinking about this task. In addition to answering the question about the behavior of the top of the ladder, prompts were included to promote the thinking and connections described in the covariation framework.

The fifth activity was designed to adhere to the six principles of model-eliciting activities (Lesh et al., 2000). This activity prompted students to provide a graph to represent the height as a function of volume for animating the filling of a container on a screen, and to produce a manual for a company to animate the

filling of any shaped container. The reason for asking students to produce a manual was to place them in a situation where they needed to organize and verbalize their understandings with a level of clarity and detail that could be understood by others. We believe that this type of task is extremely valuable in promoting a robust and flexible understanding in students (Lesh et al., 2000).

A similar set of activities were developed for the concepts of limit and accumulation. While each activity had a primary conceptual focus on the calculus concept, whenever possible, prompts were included to encourage students to connect their newly developed calculus concept with their more informal knowledge that was acquired in the context of the covariational reasoning module. Students were provided tasks that required them to unpack their calculus understanding to explain such things as why a concave up graph conveys rate of change increasing or why an inflection point corresponds to the point where the rate changes from increasing to decreasing or decreasing to increasing. As appropriate, the problems used in the covariation module were revisited and extended in the limit and accumulation modules. The covariational reasoning abilities that emerged in these subjects over the course of one semester are reported. This is followed by two examples of how covariational reasoning was used when responding to specific limit and accumulation tasks.

Select Results

Results from a Covariation Task

Comparison of pre- and post-test data revealed positive shifts in these students' covariational reasoning abilities over the course of one semester. On a pretest assessment administered on the first day of class, students were asked to select the graph (from four possible graphs) that represented the height as a function of volume as water filled an empty spherical bottle at a constant rate. The four choices were an increasing straight line, a graph that was strictly concave up, a graph that was strictly concave down and the correct graph that was initially concave down; then concave up. Of the 24 students who completed both the pre- and post-test assessment, 13 selected the graph of the increasing straight line, 6 selected either the strictly concave up or strictly concave down graph and only 5 selected the correct graph.

On the post-test assessment (administered on the last day of class) students were asked to produce a graph of the height as a function of volume for a container with a spherical bottom, a cylindrical middle and a funnel at the top (Figure 1). They were also prompted to provide a clear description of the thinking they used to construct their graph. 23 of the 24 students provided a mostly correct response (i.e., a graph that was concave down, then concave up, followed by a linear segment, then another concave down construction). Only 8 of the 24 students had a minor error in the construction of the linear segment of their graph—the slope of the linear segment was not the same as the rate/slope of the tangent line at the point to the left of the linear portion).

Construct a graph of the height as a function of volume for the given container.
Explain the reasoning you used when constructing your graph.

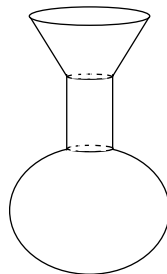


Figure 1: The Bottle Task

The written justifications for their constructions varied in form, with seven different categories of reasoning emerging from their responses. The various response types (each presented in the words of a student) are followed by the numbers of students that provided that response and a classification of that response according to the covariation framework (e.g., MA 5).

- a) For the bottom half of the bottle the height was increasing but at a decreasing rate; then after the middle, the height increased at an increasing rate (7 students) (MA 5).

- b) At first there is an increasing amount of volume needed as the height raises; then there is an increasing height as the volume changes evenly (1 student) (MA5).
- c) As the bottle gets wider, the rate it filled decreased; then as the bottle gets narrower the rate if filled increased (7 students) (MA5).
- d) As the cross-sections get progressively wider the height increases at a slower rate because it takes more volume to fill the container evenly—as the cross-sections get smaller the height increases at a quicker and quicker rate (2 students) (MA5).
- e) From the initial filling, the rate of the height increases more slowly than the volume, resulting in a concave down shape; then at the halfway point the rate of increase of the height increased more and more rapidly resulting in a concave up graph (3 students) (MA5).
- f) Longer to fill results in a concave down construction and less time to fill results in a concave up construction (2 students) (not enough information to classify).
- g) If the bottle is getting wider, it is concave down and if the bottle is getting narrower it is concave up (2 students) (not enough information to classify).

These results suggest that, at the completion of the course, most of the students in this study were reasoning at the MA5 level according to the covariation framework (i.e., they able to construct continuous images of increasing and decreasing rate while imagining the input variable changing over the entire domain). This result is particularly striking compared to previous results (Carlson, 1998) reporting that 75% of high performing second semester calculus students at the completion of the course were not able to create the correct graph for this situation. These students produced a straight line or a strictly concave up or concave down graph for this situation, while providing MA2 level reasoning for their justifications. Follow-up interviews with select students also revealed little or no ability of these students to attend to the covariant nature of this situation.

Follow Up Interviews

The follow-up interviews with eight students provided additional insights regarding the abilities of these students to access and apply covariational reasoning in a variety of situations. In addition to prompting students to provide a verbal explanation of the reasoning they used when completing the bottle task, the interview subjects were also asked to talk though the thinking they employed when responding to three additional dynamic tasks. As one example, students were asked to produce a graph of the distance (i.e., shortest distance from the starting line) as a function of time for a mile race that Tim ran at a steady pace on a quarter mile oval track. When completing this task, seven of the eight interview subjects produced an accurate construction of this situation, with their justifications again revealing diversity in their explanations, while their reasoning revealed a consistent patterns in their covariational reasoning abilities. Student responses on these tasks further confirmed that these subjects were able to effectively access and apply covariational reasoning when thinking about unfamiliar dynamic situations.

Results from A Limit Task

When asked to determine the $\lim_{x \rightarrow 2}$ for two different functions (e.g., an upside down parabola with a point removed at $x = 2$; a graph of a split function defined by $y=3$ for $x \leq 2$ and $y = 1$ for $x > 2$) most students were able to determine the limits as x approached 2 for these functions (only 2 of the 24 provided an incorrect response to the first item, and 3 of the 24 students provided an incorrect response when asked to evaluate the limit as x approaches 2 for the second function).

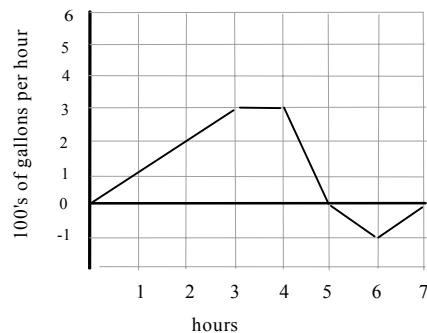
During the follow-up interview all eight students provided explanations that exhibited a consistent pattern of coordinating an image of the independent and dependent variables changing concurrently. Sue described her thinking to the first task by saying, “I thought about values of x that were approaching 2 from the left side and thought about what the y -value was getting close to, and this was 3. Then I thought about x -values getting closer and closer to 2 from the right and looked at what the y -value was getting close to while the x got closer and closer to 2. Since when x got close to 2 from both sides, y gets closer and closer to 3, this means that the limit is 3...it doesn’t matter that it never really gets to 3. The limit is what it gets close to...I mean what the y -value gets close to as x gets close to some value from both sides”. Even though one student did not provide the correct response, his difficulty appeared to result from the fact that he did not understand that y must approach the same value when x approaches 2 from both the left and the

right. This finding is not surprising as it suggests that a robust and flexible covariational reasoning ability, although foundational for this task, does not assure that students will produce a correct response, as other difficulties/misconceptions may exist.

Results from an Accumulation Task

The following accumulation task (Figure 2) was administered on the post-test assessment given on the last day of class.

Let $f(t)$ represent the rate at which the amount of water in Tempe's water reservoir changed in (100's of gallons per hour) during a 7 hour period from 10 a.m. to 5p.m. last Saturday (Assume that the tank had 5700 gallons in the tank at 10 a.m. ($t=0$)). Given the graph of f ,



- Determine how much water was in the tank at noon.
- Determine the time intervals when the water level was increasing.
- Construct a graph of g , given that $g(x) = \int_0^x f(t)dt$.
- Explain what $g(x)$ conveys about this situation.
- Determine the value of $g(7)$.

Figure 2. The Accumulation Task

On this item 21 students provided a correct response to parts b and c and 22 students provided a correct response to parts a, d, and e. The follow-up interviews suggested that these students were able to coordinate an image of the area accumulating while imagining the independent variable changing continuously. When prompted to explain the reasoning he used to construct the graph of g , Bill conveyed, “you just need to think about how much area has been added on up to that point, and then that amount gives you how much....I mean that value tells what the y -value for g is....so g gives you the amount of area that has been accumulated in the f function”. Even more surprising was the fact that 21 of the 24 students in this study were able to accurately construct the graph of g . This type of task has been previously reported to cause difficulty for students (Thompson, 1994). Further analysis of the interview responses revealed that 7 of the 8 interview subjects appeared to determine the value of the output for g by continuously coordinating an image of the accumulating area under the graph of f while imagining the time changing continuously. This result was particularly encouraging and appears to provide support for the effectiveness of their covariational reasoning abilities, and in particular their ability to coordinate images of rate accumulating while concurrently thinking about the input variable changing.

When prompted to explain their reasoning for item e above, all eight interview subjects computed the areas for contiguous sections of the graph and proceeded to explain this situation. The interview subjects each conveyed that adding the areas corresponded to the accumulation of the rate, and that the accumulation of rate for the specified time interval corresponded to the change in the amount of water in the tank from $t=0$ to $t=7$.

Concluding Remarks

These results suggest that, at the completion of the course, most of the students had developed flexible and robust covariational reasoning abilities. These results are particularly encouraging as they convey that these research based curricular materials were effective in promoting covariational reasoning in this diverse collection of first semester calculus students.

In reviewing the results for the limit and accumulation problems, it appears that these first semester calculus students were able to effectively apply their covariational reasoning abilities in a variety of contexts; in particular when thinking about conceptual tasks for two major ideas of calculus. The images that emerged in these contexts suggest that covariational reasoning may be foundational (using the term of Tall—a cognitive root) for understanding and completing specific tasks for limit and accumulation

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