CSUMS 04/07/08
PDEs in Mathematical Physics
Numerical Methods & MATLAB Implementations

http://math.asu.edu/∼bdw/CSUMS/CSUMS_04-07-08.pdf

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Convection-Diffusion Equation in 1D

- Consider the PDE

$$\frac{\partial u}{\partial t} + w\frac{\partial u}{\partial x} = \nu\frac{\partial^2 u}{\partial x^2}$$

  $(w, \nu > 0)$ with IC and BCs

$$u(x,0) = f(x), \quad u(-1,t) = u(1,t) = 0.$$

- Diffusion "smoothes" solution, Convection transports solution (wave from left to right)
- Discretization $x_0 = -1 < x_1 < \ldots < x_N < x_{N+1} = 1$ uniform

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} + w\frac{u_{i+1}^n - u_{i-1}^n}{\Delta x} = \nu\frac{u_{i+1}^{n+1} - 2u_i^{n+1} + u_{i-1}^{n+1}}{(\Delta x)^2}$$

Convection-Diffusion Equation in 1D

- Rewrite:

$$u_i^{n+1} - u_i^n + c\left(u_{i+1}^n - u_{i-1}^n\right) = d\left(u_{i+1}^{n+1} - 2u_i^{n+1} + u_{i-1}^{n+1}\right)$$

with

$$c = \frac{w\Delta t}{2\Delta x} = \text{ Courant \#}, \quad d = \frac{\nu \Delta t}{(\Delta x)^2}; \quad \frac{d}{c} = Pe = \text{ Péclet \#}$$

- Vector notation:

$$\underline{u}^{n+1} - \underline{u}^n + c\,A\underline{u}^n = -d\,B\underline{u}^{n+1}$$

with

$$\underline{u} = \begin{bmatrix} u_1 \\ \vdots \\ \vdots \\ u_N \end{bmatrix}, \quad A = \begin{bmatrix} 0 & -1 & & \\ 1 & \ddots & \ddots & \\ & \ddots & \ddots & -1 \\ & & 1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & -1 & & \\ 1 & \ddots & \ddots & \\ & \ddots & \ddots & -1 \\ & & -1 & 2 \end{bmatrix}$$

$$\Rightarrow \quad (I + dB)\underline{u}^{n+1} = (I - cA)\underline{u}^n \quad \text{with} \quad \underline{u}^0 = f(\underline{x})$$

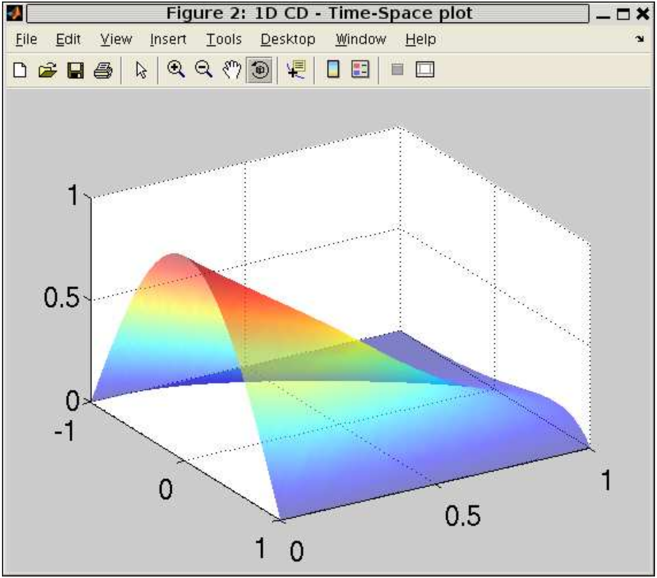


Figure 2: 1D CD - Time-Space plot
File
Edit
View
Insert
Tools
Desktop
Window
Help
1
0.5
0
-1
0
1
0
0.5
1

Exercises

- Download **cd_1d_v1.m**
  1. Run file.
  2. Rotate figure 2: manually, `view(50,30)`, `view(0,90)`
  3. Print: `print -depsc cd_1d_fig2.eps`, `print -djpeg -zbuffer cd_1d_fig2.jpg`
  4. Decrease `nu = .005` (advection dominated); reset
  5. Increase `tf = 50` and `c = 35` (unstable); reset
  6. Experiment with explicit diffusion, e.g. `M = I; N = I-d*B-c*A;` (reduce Courant number, CFL condition)
  7. Create .avi movie
- Download **cd_1d_v2.m**
  1. Compare output with fully discrete scheme from cd_1d_v1.m
  2. Compare the length of `T` with `ode15s` (stiff) and `ode45` (non-stiff)

Convection-Diffusion Equation in 2D

- Consider the PDE

$$\frac{\partial u}{\partial t} + \underline{w} \cdot \nabla u = \nu \Delta u$$

  in square $(-1, 1) \times (-1, 1)$ with IC and BCs

$$u(x, y, 0) = f(x, y, 0), \quad u(\pm 1, \pm 1, t) = f(\pm 1, \pm 1, t).$$

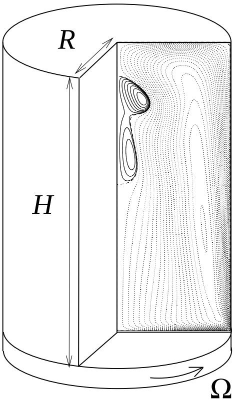
- Uniform discretization in x (N_x interior points) and y (N_y interior points)

Exercises

- Download **cd_2d_v1.m**
  1. Run file. When does solution reach “steady state”?
  2. Increase `nu = 1.5e-3` (unstable); reset
  3. Increase time-step `dt = 1.5e-3` (unstable); reset
  4. Set `w = [10 0]`, `w = [0 10]`; comment; reset
- Download **cd_2d_v2.m**
  1. Compare output with cd_2d_v1.m

Systems of PDEs: Axisymmetric Navier-Stokes

- WIKIPEDIA on Navier-Stokes
- J. M. Lopez, *Axisymmetric vortex breakdown*, JFM 221 (1990) 533–552



- Notes on problem implementation (PDF)

$$\underline{u} = (u, v, w) = \left(-\frac{\psi_z}{r}, \frac{\Gamma}{r}, \frac{\psi_r}{r}\right)$$

- ns_v1.m, ns_v2.m
- output set from actual calculation