DETECTION OF EDGES FROM NONUNIFORM FOURIER DATA

ANNE GELB AND TAYLOR HINES

ABSTRACT. Edge detection is important in a variety of applications. While there are many algorithms available for detecting edges from pixelated images or equispaced Fourier data, much less attention has been given to determining edges from nonuniform Fourier data. There are applications in sensing (e.g. MRI) where the data is given in this way, however. This paper introduces a method for determining the locations of jump discontinuities, or edges, in a one-dimensional periodic piecewise-smooth function from nonuniform Fourier coefficients. The technique employs the use of Fourier frames. Numerical examples are provided.

1. INTRODUCTION

Edge detection represents a fundamental area of signal and image reconstruction. High resolution post-processing methods for piecewise-smooth functions critically depend on correct knowledge of the edge locations. Examples of important applications for edge detection methods abound: edge detection in medical imaging (e.g. magnetic resonance (MR) imaging or computed tomography (CT)) often corresponds to differentiation between two tissue types, which is important when locating tumors or identifying other medical conditions. Edge detection is also important for post-processing PDEs, in particular when a solution contains shocks, as is encountered when modeling tsunamis, earthquakes, and similar catastrophic phenomena.

The general assumption when implementing edge detection techniques is that data is given in physical space (e.g. a pixelated image or discretized solution). As described in [28], there exist many different methods for finding the edges in a function from its discrete data, and all such methods involve essentially three steps. First, the function data is smoothed in order to remove possible artifacts near jump discontinuities as well as to reduce noise. Next, the reconstruction is differentiated numerically. Each edge detection technique may differ in how this step is performed. However, the primary objective is to detect spikes in the derivative, which correspond to a sharp change in the function value (i.e. an edge). The final step, called labeling, examines the derivative computed and identifies edge locations while suppressing false edges. Edge labeling may also be employed after a series of smoothing and differentiation procedures have been performed.

In some applications, however, data is actually collected in the frequency domain, specifically in the form of Fourier coefficients. MR or CT imaging and spectral methods for PDEs are archetypal examples of applications where this is the case. Of course, physical space edge detection techniques can be implemented on the Fourier reconstruction of the

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image, but reconstruction, filtering, and other processing introduce computational cost and approximation error, [13].

This paper does not study commonly implemented physical space edge detection techniques for several reasons. First, since we are primarily interested in the case where data is given nonuniformly and in the Fourier domain, the corresponding physical reconstruction is often not accurate, [27], and hence any physical space edge detector would perform less accurately than usual. Also, as described in [14, §5.1], the reconstruction artifacts present in nonuniform Fourier reconstructions may be exacerbated (rather than mollified) by filtering, and therefore this critical step in physical space edge detectors actually decreases overall performance. Hence it is important to investigate methods of finding edges directly from Fourier data.

There have been several successful approaches taken towards determining edge locations from Fourier data (see e.g. [9, 10, 15, 23] and references therein). In particular, the concentration factor edge detection method, first introduced in [15], accurately finds the jump locations and their associated values by constructing a modified partial Fourier sum. In short, the partial Fourier sum of a piecewise-analytic function is modified so that the energy of the partial sum approximation is “concentrated” at the singular support of the underlying function, thereby highlighting its jump discontinuities. Because it is central to the main investigation of this paper, we review this method more thoroughly in Section 2.

The edge detection method presented in [22, 23] uses Bayesian estimation to determine jump discontinuities probabilistically. This method models a function with a single jump as a continuous (nonperiodic) function $f : [-\pi, \pi] \rightarrow \mathbb{R}$ that has been rotated so that it has a discontinuity in the interior rather than the boundary of the domain. It then computes the most likely rotation given the Fourier coefficients of f . While the method has been demonstrated to accurately detect jump locations even in the presence of thin shear layers, we do not investigate its extension to nonuniform Fourier data. The method ostensibly relies on a closed formula for computing Legendre coefficients given Fourier coefficients, and we do not know of any analogous formula given nonuniform Fourier data (or even if such a formula is possible).

Finally, we note that there are some advantages in combining the Fourier-based concentration factor edge detection with minimization techniques, [24, 25, 26]. We will not discuss these ideas here, but may incorporate them into future investigations.

The general assumption in all of these edge detection methods is that the underlying function is sampled uniformly in frequency space (using the standard Fourier basis). This is not always the case, however. In recent years, practitioners and scientists have begun to investigate the use of non-Cartesian sampling (in the frequency domain) for acquisition of MRI data in order to decrease acquisition time and reduce motion artifacts, [1, 20]. Further, in any application where data is collected mechanically, the samples may be jittered from the (presumably) uniform case, thus destroying the orthogonality of the corresponding basis functions.

One possible solution, called uniform resampling, generates uniform Fourier coefficients by interpolating the given nonuniform Fourier data. Another technique, often referred to as convolutional gridding with density compensation, improves the nonuniform IFFT by ‘compensating’ for the variable density in the frequency domain. The method simultaneously regrids the data via convolution so that the fast Fourier transform (FFT) can be used. For a

detailed investigation into these and related methods, see [8, 11, 20, 26, 27]. Unfortunately, these types of methods all have drawbacks, including interpolation error and ill-conditioned numerical algorithms.

In this paper we adopt a different strategy. Instead of interpolating to uniform Fourier coefficients, we extend the concentration factor edge detection method, [15], to work directly on the given nonuniform Fourier data. Hence no interpolation error is incurred, and the numerical algorithm remains well-conditioned.

This paper is organized as follows. In Section 2 we review the concentration factor edge detection method for approximating the jump function corresponding to a piecewise-smooth function from given uniform Fourier coefficients, [15, 16]. In Section 3 we introduce Fourier frames as a way of investigating nonuniform Fourier data. We also discuss some commonly encountered nonuniform sampling schemes that are of particular interest in practical applications. In Section 4 we demonstrate how Fourier frame theory can be implemented using finite-dimensional frame approximations. Section 5 extends the concentration factor edge detection method to Fourier frames and presents the corresponding numerical analysis. Computational results for several examples using the frame-theoretic edge detection method are given in Section 6. We compare the performance and cost analysis of our new method to the standard concentration factor edge detection method using interpolated Fourier coefficients in Section 7. Section 8 provides a few concluding remarks.

2. CONCENTRATION FACTOR EDGE DETECTION

2.1. Background. Consider a periodic, piecewise-smooth function $f : [-\pi, \pi) \rightarrow \mathbb{R}$. Define the associated *jump function* $[f] : (-\pi, \pi) \rightarrow \mathbb{R}$ by

$$(1) \quad [f](x) = \lim_{t \downarrow x} f(t) - \lim_{t \uparrow x} f(t).$$

Notice that $[f]$ is well-defined since f is piecewise-continuous. Furthermore, it is easy to see that $[f] = 0$ everywhere except where f has a jump discontinuity, in which case $[f]$ takes the value of the jump.

If f has jump locations $\xi_i \in (-\pi, \pi)$, $1 \leq i \leq m$, and we denote by $[f](\xi_i)$ the value of the jump at ξ_i , then the jump function can also be defined by

$$(2) \quad [f](x) = \sum_{i=1}^m [f](\xi_i) \mathbb{I}_{\xi_i}(x),$$

where $\mathbb{I}_{\xi}$ is the indicator function at ξ . This will be a convenient alternative formulation of $[f]$ when nonuniform data is used. Note that the two definitions, (1) and (2), are equivalent. Our goal is to recover the jump function $[f]$ as accurately as possible from a finite number of Fourier coefficients.

Define $\hat{f} : \mathbb{R} \rightarrow \mathbb{C}$ as the Fourier transform of f ,

$$(3) \quad \hat{f}(n) = \int_{-\pi}^{\pi} f(x) e^{inx} dx.$$

The celebrated Shannon-Nyquist sampling theorem shows that the original function f can be reconstructed exactly given the value of $\hat{f}$ on $\mathbb{Z}$. That is, the partial Fourier series, given

by the operator S_N , has the property that

$$(4) \quad S_N f = \sum_{n=-N}^N \hat{f}(n) e^{inx} \rightarrow f \quad \text{as } N \rightarrow \infty.$$

Although this series converges exponentially fast for periodic analytic functions, if f is piecewise-analytic (or nonperiodic), the partial Fourier approximation is plagued by reconstruction artifacts ($\mathcal{O}(1)$ oscillations near jump discontinuities and $\mathcal{O}(1/N)$ accuracy overall), collectively known as the Gibbs phenomenon, [2, 3, 17]. However, while this poor response to jump discontinuities is bad for reconstruction purposes, it is evidence that the Fourier coefficients of a piecewise-analytic function contain information that can be used for edge detection. Specifically, the presence of the artificial oscillations at the jump discontinuities of f indicate where, in fact, the discontinuities are located, [9, 15, 22].

One of the first results that demonstrates this fact, due to Lukács (see e.g. [9, p. 507], [15, Thm. 2.1, p. 108]), shows that the modified Fourier sum

$$(5) \quad \sum_{n=-N}^N \left(\frac{-\pi i \operatorname{sgn}(n)}{\log N} \right) \hat{f}(n) e^{inx} \rightarrow [f](x) \quad \text{as } N \rightarrow \infty.$$

However, this series converges as $\mathcal{O}(1/\log N)$, too slowly for most practical purposes. Nevertheless, this result motivated a host of similar edge detection techniques using Fourier data. In particular, if a periodic function f with $f(\pi) = f(-\pi)$ has a simple jump discontinuity at $\xi \in (-\pi, \pi)$, then the following relation between the Fourier coefficients and the jump discontinuity using integration by parts is easily derived:

$$(6) \quad \begin{aligned} \hat{f}(n) &= \int_{-\pi}^{\pi} f(x) e^{-inx} dx \\ &= \int_{-\pi}^{\xi^-} f(x) e^{-inx} dx + \int_{\xi^+}^{\pi} f(x) e^{-inx} dx \\ &= \frac{[f](\xi) e^{-in\xi}}{in} + \int_{-\pi}^{\pi} f'(x) \frac{e^{-inx}}{-in} \\ &\approx \frac{[f](\xi) e^{-in\xi}}{in} + \mathcal{O}(1/n^2). \end{aligned}$$

The relationship in (6) is readily extended for multiple jump discontinuities in $(-\pi, \pi)$.

Using only information from the given Fourier coefficients, the edge detection method described in [15] modifies the partial Fourier series, (4), so that it converges to the jump function, (1). More specifically, a *concentration factor* is any function $\sigma : [-1, 1] \rightarrow \mathbb{C}$ designed so that

$$(7) \quad S_N^\sigma f = \sum_{n=-N}^N \sigma\left(\frac{n}{N}\right) \hat{f}(n) e^{inx} \rightarrow [f](x) \quad \text{as } N \rightarrow \infty.$$

We will refer to (7) as the concentration factor method for edge detection. In [15], the authors give a more practical characterization of concentration factors in the following result:

Theorem 1 (Admissible concentration factors). [15, Thm. 3.1, p. 120] *Consider the odd conjugate kernel*

$$\tilde{K}_N^\sigma = -\frac{1}{\pi} \sum_{k=1}^N \sigma(k/N) \sin(kt)$$

associated with a $C^2[0, 1]$ concentration function σ , such that $|\sigma(1/N)| \leq \mathcal{O}(1/\log N)$. If

$$(8) \quad \int_{1/N}^1 \frac{\sigma(x)}{x} dx \rightarrow -\pi \quad \text{as } N \rightarrow \infty, \text{ and}$$

$$(9) \quad \sum_{k=1}^N \frac{|\sigma(k/N)|}{k^2} \rightarrow 0 \quad \text{as } N \rightarrow \infty$$

then σ satisfies the property that

$$S_N^\sigma f = f * \tilde{K}_N^\sigma \rightarrow [f] \quad \text{as } N \rightarrow \infty.$$

In this same paper, the authors perform an in-depth analysis of the accuracy and convergence rates of different concentration factors. We direct the reader to [15, p.121] for this information. A concentration factor can be viewed as the dual idea to a spectral filter. In fact, the characterizations are almost identical other than the fact that a filter emphasizes low frequencies and mollifies high frequencies, while a concentration factor must include high frequency information. In its general form, σ has a windowing effect that highlights the edges of a discontinuous function. Three different concentration factors presented in [16] are displayed in Figure 1:

i. Trigonometric factor, [16, p.1395],

$$(10) \quad \sigma^{trig}(x) = \frac{i\pi \sin \pi x}{\text{Si}(\pi)}$$

where $\text{Si}(\pi) = \int_0^\pi \frac{\sin t}{t} dt$.

ii. Polynomial factor, [16, p.1395],

$$(11) \quad \sigma^{poly}(x) = i\pi p x^p$$

for positive integer p .

iii. Exponential factor, [16, p.1397],

$$(12) \quad \sigma^{exp}(x) = i\pi C x e^{\frac{1}{\alpha x(x-1)}}$$

where $C = \int e^{\frac{-1}{\alpha t(t-1)}} dt$, and $\alpha > 0$.

In all of the numerical results presented here, we use $p = 1$ and $\alpha = 5$. Figure 2 demonstrates the ability of the concentration factor method, (7), to compute $[f]$, (1), for the following example:

Example 1. Define the function $f : [-\pi, \pi] \rightarrow \mathbb{R}$ by

$$(13) \quad f(x) = \begin{cases} \sin 7x & \text{if } x < -\pi/2 \\ -\sin 2x & \text{if } x \geq -\pi/2. \end{cases}$$

Figure 1: The imaginary parts of σ^{trig} , (10), σ^{poly} with $p = 1$, (11), and σ^{exp} with $\alpha = 5$, (12).

Figure 2: The concentration factor method, (7), applied to Example 1. Here we use concentration factors (10) - (12) and $N = 128$.

As is evident from Figure 2, the various concentration factors lead to different convergence rates away from the jump discontinuities and yield different oscillatory patterns as each jump discontinuity is approached. In particular, the ‘low order’ methods, which use (10) and (11), do not suffer from oscillations in the regions of discontinuities, but they are typically unable

Figure 3: The concentration factor method, (7), with minmod enhancement, (14), applied to Example 1. Here we use concentration factors (10)–(12) and $N = 128$.

to resolve the jump function in smooth regions. On the other hand, the ‘high order’ method, which uses (12), converges quickly to zero away from jump discontinuities but has strong oscillations near them. Unfortunately, this type of behavior means that standard thresholding techniques for detecting edges directly from (7) may fail. This difficulty was addressed in [12], where nonlinear enhancement techniques were introduced to better pinpoint the edge locations. Specifically, a better approximation of $[f](x)$ can be obtained by combining the results of (7) using several different concentration factors as

$$(14) \quad [f](x) \approx MM(S_N^{\sigma_1} f(x), S_N^{\sigma_2} f(x), \dots, S_N^{\sigma_n} f(x)).$$

Here $MM : \mathbb{R}^n \rightarrow \mathbb{R}$ is the *minmod* function defined by

$$(15) \quad MM(a_1, \dots, a_n) = \begin{cases} s \min\{|a_1|, \dots, |a_n|\} & \text{if } \text{sgn } a_1 = \dots = \text{sgn } a_n = s \\ 0 & \text{otherwise.} \end{cases}$$

Figure 3 demonstrates the use of (14) on Example 1 using the trigonometric, (10), polynomial, (11), and exponential, (12), concentration factors. Standard thresholding techniques can then be applied to determine the exact jump locations. More discussion on the use of enhancement techniques to recover the jump locations can be found in [12, 16].

2.2. The Concentration Factor Method Applied to Nonuniform Fourier Data. As a naive approach, let us consider directly applying (7) to nonuniform Fourier data:

$$\sum_{n=-N}^N \sigma(n/N) \hat{f}(\lambda_n) e^{i\lambda_n x}.$$

As can be seen in Figure 4, the results do not converge to the corresponding jump function of the saw tooth function. This is due to the fact that Theorem 1 is no longer valid for

Figure 4: The concentration factor method, (7), applied directly to nonuniform Fourier samples of the sawtooth function, (Example 2), using jittered sampling, (21) with (23), (left), and log-spaced sampling, (24), (right). Both figures use $N = 64$.

nonuniform sampling because the so-called “conjugate kernel” in this case does not converge to $\tilde{K}_N^\sigma$, (see e.g. [27]).

It is possible to interpolate the Fourier space data in order to apply edge detection methods based on uniform Fourier coefficients, [26]. However, the ill-conditioning of the corresponding interpolation matrix causes the interpolated high frequency coefficients to be inaccurate, making the concentration factor method less robust with respect to the sampling scheme.¹ This will be discussed further in Section 6. In the next section we introduce another approach using frame theory that will enable direct application of the given nonuniform Fourier data for edge detection.

3. FOURIER FRAMES

In this section we collect some results from frame theory that will be useful in extending the concentration factor method to the nonharmonic case. In particular, although the purpose of this investigation is not to reconstruct the underlying piecewise-smooth function from its nonharmonic partial sum (see [14] for methods that do this), it is clear from the discussion of the concentration factor method in Section 2, and more precisely, (7), that formulating the reconstruction process is critical to developing a suitable edge detection algorithm.

3.1. General theory. It is well-known that the family $\{e^{inx}\}_{n \in \mathbb{Z}}$ forms an orthonormal basis for the Hilbert space $L^2[-\pi, \pi]$, but this is not the case for more general families of exponentials. Frames of exponentials (or *Fourier frames*) were introduced in [7] as a way of extending the techniques of standard Fourier analysis to the nonharmonic case.

Definition 1 (The frame condition). Given an interval $I \subset \mathbb{R}$ and a sequence $\{\lambda_n\}_{n \in \mathbb{Z}}$ of real numbers, we say that the family $\{e^{i\lambda_n x}\}_{n \in \mathbb{Z}}$ is a (Fourier) frame for $L^2(I)$ if, for constants

¹The amount of ill-conditioning is directly related to the irregularity of the sampling scheme in Fourier space, [26].

$0 < A \leq B$, it satisfies the *frame condition*:

$$(16) \quad A\|f\|^2 \leq \sum_{n \in \mathbb{Z}} |\langle f(x), e^{i\lambda_n x} \rangle|^2 \leq B\|f\|^2$$

for all $f \in L^2(I)$.

The lower bound in the frame condition forces the family $\{e^{i\lambda_n x}\}$ to be complete in $L^2(I)$, while the upper frame condition ensures that the sequence of coefficients $\{\langle f(x), e^{i\lambda_n x} \rangle\} \in \ell^2(\mathbb{Z})$, the two conditions necessary to reasonably address synthesis and analysis. Note that a Fourier frame is not in general orthonormal, or even linearly independent. It is often repeated that a convenient description of a frame is as an “overcomplete basis.”

If $\{e^{i\lambda_n x}\}_{n \in \mathbb{Z}}$ is a frame for $L^2(I)$, then the associated *frame operator* $S : L^2(I) \rightarrow L^2(I)$ can be defined by

$$(17) \quad Sf = \sum_{n \in \mathbb{Z}} \langle f(x), e^{i\lambda_n x} \rangle e^{i\lambda_n x}.$$

Theorem 2 (The frame operator). [7] *Any frame operator is bounded, invertible, positive, and self-adjoint.*

As an immediate result of Theorem 2, we obtain (by self-adjointness) the identity

$$(18) \quad f = SS^{-1}f = \sum_{n \in \mathbb{Z}} \langle f(x), S^{-1}e^{i\lambda_n x} \rangle e^{i\lambda_n x}$$

which shows that f can be represented (or analyzed) by the sequence $\{\langle f(x), S^{-1}e^{i\lambda_n x} \rangle\}_{n \in \mathbb{Z}}$ in $\ell^2(\mathbb{Z})$. Similarly, we obtain

$$(19) \quad f = S^{-1}Sf = \sum_{n \in \mathbb{Z}} \langle f(x), e^{i\lambda_n x} \rangle S^{-1}e^{i\lambda_n x}$$

which demonstrates how f can be reconstructed (or synthesized) exactly from its frame coefficients.

A fundamental result about frames is that if $\{e^{i\lambda_n x}\}$ is a frame for $L^2(I)$ with bounds $A \leq B$, then the functions $\{S^{-1}e^{i\lambda_n x}\}$ form a frame for $L^2(I)$ with bounds $1/B, 1/A$, called the (*canonical*) *dual frame*, [5]. Furthermore, it follows that S^{-1} is the frame operator associated with the dual frame and

$$(20) \quad 1/B \leq \|S^{-1}\| \leq 1/A.$$

As shown in (19), every function can be expanded as a sum of dual frame functions if the frame coefficients are known. However, this expansion will not be unique in general. In fact, there are usually many different ways of writing out the frame expansion of a function since the frame functions (as well as the dual frame functions) are not linearly independent. Furthermore, there is no closed form equation for constructing the inverse frame operator S^{-1} , which makes the theoretical result in (18) or (19) computationally infeasible. For this reason, several methods have been developed to numerically construct a finite-dimensional frame approximation, which will be further discussed in Section 4.

3.2. Sampling sequences. Although there exists a wide array of sampling sequences which generate frames, we will focus on a couple of particular cases. The first is the jittered sampling scheme, of the form

$$(21) \quad \lambda_n = n \pm \eta,$$

where η is a random variable. This sampling scheme is of particular interest when investigating reconstruction techniques in the presence of machine error that causes samples to be collected with slight irregularities. We consider the following distributions:

$$(22) \quad \eta \text{ uniformly distributed on } [0, 1/4]$$

$$(23) \quad \eta \text{ uniformly distributed on } [0, 1/2]$$

The second is the log-spaced sampling scheme,

$$(24) \quad \{\lambda_n\} \text{ spaced logarithmically, centered around 0.}$$

Figure 5 illustrates these sampling patterns. Note that the log-spaced sampling scheme does not form a frame, since the sparse sampling at high Fourier modes prevents the corresponding sampling functions from spanning. This reflects particularly poorly on edge detection algorithms, since edge information is primarily contained in the high frequency samples. However, the log-spaces scheme is of interest because it is the one-dimensional analogue of a spiral, which is a commonly-studied sampling pattern in MR imaging (see e.g. [1]).

Figure 5: An example of different Fourier sampling sequences $\{\lambda_n\}_{n=-16}^{16}$ as compared to standard (uniform) Fourier sampling.

4. FINITE-DIMENSIONAL FRAME RECONSTRUCTION

4.1. The frame algorithm. In practice, we only deal with a finite number of frame functions, and hence we must be satisfied with a finite-dimensional approximation. Consider the operator

$$(25) \quad S_N f = \sum_{n=-N}^N \langle f(x), e^{i\lambda_n x} \rangle e^{i\lambda_n x}$$

which, if this frame is the standard Fourier basis, is a direct generalization of the partial Fourier sum, (4). First, notice that if $\{e^{i\lambda_n x}\}_{n \in \mathbb{Z}}$ is a frame for $L^2(I)$ then $\{e^{i\lambda_n x}\}_{n=-N}^N$ is a frame for its (linear) span.² It is not hard to see that the frame operator associated

²In fact, it is straightforward to show that *any* finite family of functions is a frame for its span (see [5]).

with this finite subfamily is exactly (25). Therefore it follows that S_N is invertible (on $\text{span}\{e^{i\lambda_n x}\}_{n=-N}^N$) and that $S_N \rightarrow S$ uniformly as $N \rightarrow \infty$ (see [6], for example). We call the operator S_N the *finite frame operator* associated with the frame $\{e^{i\lambda_n x}\}_{n \in \mathbb{Z}}$. A theorem in [4, Thm. 3, p.83] proves that $S_N^{-1} \rightarrow S^{-1}$ strongly³ as $N \rightarrow \infty$, and hence finite-dimensional approximations of the frame operator are possible.

In particular, we can see from (25) that the operator

$$(26) \quad T_N f = \sum_{n=-N}^N \langle f(x), e^{i\lambda_n x} \rangle S_N^{-1} e^{i\lambda_n x},$$

called the *finite frame approximation*, is exactly the projection of f onto $\text{span}\{e^{i\lambda_n x}\}_{n=-N}^N$. Once again we note that the inverse (finite) frame operator S_N^{-1} is computationally infeasible to construct. Fortunately, an iterative algorithm was given in [7] for calculating these expansions provided only with the frame coefficients.

Theorem 3 (The Frame Algorithm). [7] *Let $\{e^{i\lambda_n x}\}_{n \in \mathbb{Z}}$ be a (Fourier) frame for $L^2(I)$ with bounds $A, B > 0$, and let $f \in L^2(I)$. Iteratively define the functions f_k , $k \in \mathbb{N}$ by*

$$\begin{aligned} f_0 &= 0 \\ f_k &= f_{k-1} + \frac{2}{A+B} S(f - f_{k-1}) \end{aligned}$$

for $k \geq 1$. Then $f_k \rightarrow f$ with error bounds

$$\|f_k - f\| \leq \left(\frac{B-A}{B+A} \right)^k \|f\|.$$

The proof of this theorem essentially relies on the fact that S is positive and that $\|I - S\| \leq \frac{B-A}{B+A} \leq 1$, which follows from the frame condition, (16), and Theorem 2. It is an immediate corollary that the frame algorithm can be applied in the finite-dimensional case as well, by simply replacing S with S_N . In this case, we have $f_k \rightarrow T_N f$ as $\|f_k - T_N f\| \leq \left(\frac{B_N - A_N}{B_N + A_N} \right)^k \|T_N f\|$, where A_N and B_N are the respective lower and upper bounds for the frame $\{e^{i\lambda_n x}\}_{n=-N}^N$. Hence the frame algorithm recovers the approximation (26).

Note that if in a particular application the sampling scheme is known a priori, then the frame algorithm can be used to calculate the dual frame functions. By setting $f_1 = \frac{2}{A+B} e^{i\lambda_n x}$ for fixed $n \in \mathbb{Z}$, it is easy to see that $f_k \rightarrow S^{-1} e^{i\lambda_n x}$. Therefore the dual frame can be computed off line in many applications, which greatly decreases the effective computational cost in these cases. Notice also that knowledge of the frame bounds is important, as the convergence of the algorithm is sensitive to the ratio A/B , with $A/B \approx 0$ resulting in very slow convergence and $A/B \approx 1$ resulting in fast convergence. Finally, several accelerated frame algorithms, developed in [18], can vastly improve the convergence of the frame algorithm when the frame bounds are poor. The (Chebyshev) accelerated frame algorithm [18, Thm. 1, p. 3333] improves the convergence rate of the relative error to $\left(\frac{\sqrt{B}-\sqrt{A}}{\sqrt{B}+\sqrt{A}} \right)^k$. All of our numerical results use this algorithm.

³That is, in the strong operator topology. For more details on finite-dimensional approximations of the frame operator, see [4].

4.2. Convergence of finite-dimensional frame reconstruction. While the frame algorithm guarantees convergence in the L^2 sense, it says nothing about pointwise convergence. Consider the following example:

Example 2. Define $f : [-\pi, \pi] \rightarrow \mathbb{R}$ by

$$(27) \quad f(x) = \begin{cases} x + \pi & \text{if } x < 0, \\ x - \pi & \text{if } x \geq 0. \end{cases}$$

Figure 6 illustrates the use of Theorem 3 to approximate Example 2 from jittered Fourier frame samples, (21) with (22). The reconstruction exhibits oscillations at the jump discontinuity, and slow convergence elsewhere, reminiscent of the Gibbs phenomenon.⁴ The presence of oscillations near the jump discontinuity indicates that the frame coefficients contain information about the jump discontinuities of the sampled function just as in the uniform case, providing further motivation for extending the concentration factor method to the nonharmonic case.

Figure 6: The frame reconstruction, (26), of Example 2 using the jittered Fourier samples, (21) with (22). Here we use $N = 32, 64, 128$, and 256 .

⁴We chose an example with zero boundaries so as to not confuse the issue of approximation error (that may or may not be a periodicity issue) versus edge error which is clearly a Gibbs-related artifact. Approximation issues and results for removing such artifacts from frame reconstructions can be found in [14].

5. CONCENTRATION FACTORS FOR NONUNIFORM FOURIER DATA

Let $\{e^{i\lambda_n x}\}_{n \in \mathbb{Z}}$ be a frame for $L^2[-\pi, \pi]$, and let $f \in L^2[-\pi, \pi]$ be periodic and piecewise smooth, with $f^{(m)}(\pm\pi) = 0$ for all $m \geq 0$. Furthermore, assume that the only information we are given about the function is a finite number of frame coefficients

$$(28) \quad \{\hat{f}_n\}_{n=-N}^N = \{\langle f(x), e^{i\lambda_n x} \rangle\}_{n=-N}^N.$$

We now derive the relation between the frame coefficients and the jump function just as in (6). Again for simplicity we assume the existence of a single jump discontinuity at $x = \xi$, noting that the corresponding analysis is easily extended for multiple jumps.

$$(29) \quad \begin{aligned} \hat{f}(\lambda_n) &= \frac{[f](\xi)e^{-i\lambda_n \xi}}{i\lambda_n} + \int_{-\pi}^{\pi} f'(x) \frac{e^{-i\lambda_n x}}{-i\lambda_n} \\ &\approx \frac{[f](\xi)e^{-i\lambda_n \xi}}{i\lambda_n}. \end{aligned}$$

Here, however, we cannot conclude that $\int_{-\pi}^{\pi} f'(x) \frac{e^{-i\lambda_n x}}{-i\lambda_n} = \mathcal{O}(\frac{1}{n^2})$ as in (6), nor do we know in general how this term decays for the nonuniform case.

Recall that (7) provides a technique to construct the jump function (1) from uniform coefficients. Our objective is to use this approximation to design a concentration factor $\sigma : \mathbb{R} \rightarrow \mathbb{C}$ such that the frame expansion

$$(30) \quad T_N^\sigma f = \sum_{n=-N}^N \sigma(\lambda_n) \hat{f}(\lambda_n) S^{-1} e^{i\lambda_n x} \rightarrow [f](x) \text{ as } N \rightarrow \infty.$$

We will refer to (30) as the frame-theoretic edge detection method or the frame based concentration factor method. Since $\{e^{i\lambda_n x}\}$ is a frame for $L^2[-\pi, \pi]$, the frame representation, (19), for $[f](x)$ is

$$[f](x) = \sum_{n \in \mathbb{Z}} \langle [f](x), e^{i\lambda_n x} \rangle S^{-1} e^{i\lambda_n x}.$$

However, since $[f](x)$ is zero a.e., it is necessary to regularize the indicator function $\mathbb{I}_\xi$ from (2).⁵ Furthermore, $\mathbb{I}_\xi$ must be regularized so that no a priori information about the jump function (e.g. the location ξ or the value $[f](\xi)$) is used in the resulting concentration factor σ . Figures 7 and 8 show two possible regularizations – a Gaussian distribution of small variation and a narrow hat function, respectively. In general, the shape of the regularized indicator function represents a trade-off between resolution near the jump location and convergence to zero in the smooth regions.

Once the regularized $\mathbb{I}_\xi$ is chosen, the approximation in (29) determines that the convergence of the nonuniform concentration factor expansion (30) will hold if

$$(31) \quad \sigma(\lambda_n) = \frac{i\lambda_n \langle \mathbb{I}_\xi(x), e^{i\lambda_n x} \rangle}{e^{-i\lambda_n \xi}} = i\lambda_n e^{i\lambda_n \xi} \int_{-\pi}^{\pi} \mathbb{I}_\xi(x) e^{-i\lambda_n x} dx = i\lambda_n e^{i\lambda_n \xi} \widehat{\mathbb{I}_\xi}(\lambda_n) \text{ for all } n.$$

In the following subsections we show several examples of regularized indicator functions that yield corresponding concentration factors which are independent of the jump location ξ and lead to the convergence of the frame-theoretic edge detection method, (30).

⁵For convenience we use (2) to define $[f](x)$.

5.1. Gaussian concentration factor. It is clear from the uniform case that we need the concentration factor σ to satisfy $(T_N^\sigma \mathbb{I}_\xi)(\xi) \rightarrow 1$ as $N \rightarrow \infty$, and $(T_N^\sigma \mathbb{I}_\xi)(x) \rightarrow 0$ everywhere else, where T_N^σ is defined by (30). We regularize $\mathbb{I}_\xi$ as a Gaussian distribution with mean ξ and variance ϵ ,

$$(32) \quad \mathbb{I}_\xi^G(x) = \exp\left(-\frac{(x-\xi)^2}{2\epsilon^2}\right).$$

Then, as shown in (31), we compute the transform

$$\begin{aligned} \sigma^G(\lambda_n) &= i\lambda_n e^{i\lambda_n \xi} \widehat{\mathbb{I}_\xi^G}(\lambda_n) \\ &= i\lambda_n e^{i\lambda_n \xi} \int_{-\pi}^{\pi} e^{-\frac{(x-\xi)^2}{2\epsilon^2}} e^{-i\lambda_n x} dx \\ &= i\lambda_n e^{i\lambda_n \xi} \left(\sqrt{\pi/2\epsilon} e^{-i\lambda_n \xi} e^{-\frac{1}{2}\epsilon^2 \lambda_n^2} \right) \operatorname{erf}\left(\frac{\sqrt{2}(x+\xi-i\lambda_n \epsilon^2)}{2\epsilon}\right) \Big|_{-\pi}^{\pi} \\ &= i\lambda_n \epsilon \sqrt{\pi/2} e^{-\frac{1}{2}\epsilon^2 \lambda_n^2} \left(\operatorname{erf}\left(\frac{\sqrt{2}(\pi+\xi-i\lambda_n \epsilon^2)}{2\epsilon}\right) + \operatorname{erf}\left(\frac{\sqrt{2}(\pi-\xi+i\lambda_n \epsilon^2)}{2\epsilon}\right) \right) \end{aligned}$$

where $\operatorname{erf}(z)$ is the Gaussian error function. Now, assuming that $|\xi - \pi| > \epsilon$ (which is reasonable if f is periodic), we have that $\operatorname{Re}\left(\frac{\sqrt{2}(\pi \pm (\xi - i\lambda_n \epsilon^2))}{2\epsilon}\right)$ is large, and $\operatorname{Im}\left(\frac{\sqrt{2}(\pi \pm (\xi + i\lambda_n \epsilon^2))}{2\epsilon}\right)$ is very small. Therefore,

$$\operatorname{erf}\left(\frac{\sqrt{2}(x+\xi-i\lambda_n \epsilon^2)}{2\epsilon}\right) \Big|_{-\pi}^{\pi} \approx 2$$

and we obtain the concentration factor

$$(33) \quad \sigma^G(\lambda_n) = i\lambda_n \epsilon \sqrt{\pi/2} e^{-\frac{1}{2}\epsilon^2 \lambda_n^2}.$$

In general, smaller values of ϵ correspond to better resolution (narrower regularization) but slower convergence to zero in smooth regions, just as in the uniform case.

a. Gaussian-regularized indicator function, $\mathbb{I}_\xi^G$, with $\xi = 0$ b. Gaussian concentration factor, σ^G

Figure 7: (a) An example of a Gaussian-regularized indicator function, (32), and (b) the imaginary part of the corresponding concentration factor, (33). Various values of ϵ are displayed in both figures.

5.2. Linear concentration factor. Another option is to regularize the indicator function $\mathbb{I}_\xi$ with the hat function

$$(34) \quad \mathbb{I}_\xi^L(x) = \begin{cases} \frac{1}{\epsilon}(x - \xi + \epsilon) & \text{if } \xi - \epsilon \leq x \leq \xi \\ -\frac{1}{\epsilon}(x - \xi - \epsilon) & \text{if } \xi \leq x \leq \xi + \epsilon \\ 0 & \text{else.} \end{cases}$$

The transformation is straightforwardly computed as

$$(35) \quad \begin{aligned} \sigma^L(\lambda_n) &= i\lambda_n e^{i\lambda_n \xi} \widehat{\mathbb{I}_\xi^L}(\lambda_n) \\ &= i\lambda_n e^{i\lambda_n \xi} \int_{-\pi}^{\pi} \mathbb{I}_\xi(x) e^{-i\lambda_n x} dx \\ &= \frac{i}{\lambda_n \epsilon} (2 - e^{i\lambda_n \epsilon} - e^{-i\lambda_n \epsilon}) \\ &= \frac{2i}{\lambda_n \epsilon} (1 - \cos \lambda_n \epsilon) \\ &= 2i \frac{\sin^2 \lambda_n \epsilon / 2}{\lambda_n \epsilon / 2} \end{aligned}$$

a. Linearly-regularized indicator function, $\mathbb{I}_\xi^L$ with $\xi = 0$ b. Linear concentration factor, σ^L

Figure 8: (a) An example of a linearly-regularized indicator function, (34), and (b) the imaginary part of the corresponding concentration factor, (35), with various values of ϵ .

5.3. Other concentration factors. Figure 9 depicts some analogous concentration factors to those found in [15], which are designed to mimic the form of the concentration factors given above and in [15], without the same derivation process of regularizing $\mathbb{I}_\xi$. While an analogous theorem of admissible concentration factors is desirable, it is clear that the admissibility condition would be highly dependent on the sampling scheme.

⁶The Cauchy concentration factor is derived by regularizing the indicator function $\mathbb{I}_\xi$ as a Cauchy distribution with mean ξ . The derivation of the resulting concentration factor is performed just as in Section 5.1.

Figure 9: Other concentration factors (imaginary parts)

6. RESULTS

We now provide several numerical examples to demonstrate the effectiveness of our frame-theoretic edge detection method, (30).

Figure 10 shows the results of (30) applied to Example 2 on jittered samples, (21), using different concentration factors. Note that varying the parameter ϵ in the Gaussian or linear factors (in essence, changing the width of the indicator function regularization) will produce slightly different results, but in effect is just a trade-off between resolution near the jump location and convergence speed in the smooth regions.

Figure 11 illustrates how the concentration factors behave with respect to different sampling schemes, using the jittered, (21) with both (22) and (23), and log-spaced samples, (24), for Example 2. As stated in Section 3.2, although the log-spaced scheme is not a frame, we investigate it because of practical interest.

Example 3. Define $f : [-\pi, \pi] \rightarrow \mathbb{R}$ by

$$(36) \quad f(x) = \begin{cases} \text{atan}(\pi x) - \frac{\text{atan} \pi^2}{\pi} x & \text{if } x < 1 \\ \frac{x}{2\pi} - \frac{1}{2} & \text{if } x \geq 1. \end{cases}$$

Figure 10: (left) The frame-theoretic edge detection method, (30), applied to Example 2; (right) a closeup of the same plot. Here we use jittered sampling, (21) with (23), and $N = 64$.

Figure 11: The frame-theoretic edge detection method, (30), applied to Example 2, using linear concentration factors (left) and Gaussian concentration factors (right). In each case we use jittered sampling, (21) with (23) and (22), and log-spaced sampling, (24), with $N = 64$.

Example 3 is used to demonstrate how the frame based concentration method, (30), detects edges in a function where the jump is obscured by a tall, thin shear layer, [23]. Figures 12 and 13 display the results.

Example 4. Define $f : [-\pi, \pi] \rightarrow \mathbb{R}$ by

$$(37) \quad f(x) = \begin{cases} 3/2 & \text{if } -3/4 \leq x < -1/2 \\ 7/4 - \pi x/2 + \sin(\pi x - 1/4) & \text{if } -1/4 \leq x < 1/8 \\ 11\pi x/4 - 5 & \text{if } 3/8 \leq x < 3/4. \end{cases}$$

Example 4 demonstrates how the frame based concentration factor method locates multiple jump discontinuities. These results are displayed in Figures 14 and 15.

All of these figures indicate that it is necessary to implement post-processing techniques in order to remove the spurious oscillations near the jump discontinuities and to improve the convergence in smooth regions. Once again we employ the *minmod* algorithm, (14). Figures

Figure 12: The frame based concentration method, (30), for Example 3. Here we use jittered sampling, (21) with (23), with $N = 64$ (left) and $N = 128$ (right).

Figure 13: The frame based concentration method, (30), for Example 3. Here we use log-spaced sampling, (24), with $N = 64$ (left) and $N = 128$ (right).

Figure 14: The frame based concentration method, (30), applied to Example 4. Here we use jittered sampling, (21) with (23), with $N = 64$ (left) and $N = 128$ (right).

Figure 15: The frame based concentration method, (30), applied to Example 4. Here we use log-spaced sampling, (24), with $N = 64$ (left) and $N = 128$ (right).

Figure 16: Minmod enhancement, (14), using σ^G , σ^L , σ^T and σ^E , for Examples 2, (left), and 4, (right). Here we use jittered sampling, (21) with (23), with $N = 64$ and $N = 128$.

16 and 17 demonstrate these results using the σ^G , (33), σ^L , (35), σ^T , (Figure 9c.), and σ^E , (Figure 9d.).

As expected, the sampling trajectory $\{\lambda_n\}_{n \in \mathbb{Z}}$ determines the quality of the edge detection results. If our frame emulates the standard Fourier basis (for example, in the case of jittered sampling) then we see much faster convergence to the jump function (in terms of frame coefficients needed), as well as an absence of oscillations in the smooth regions of the signal.

7. COMPARISON TO RESULTS USING RESAMPLED FOURIER COEFFICIENTS

Here we briefly compare the performance and cost analysis of our new frame-theoretic edge detection method, (30), to the standard concentration factor method, (7), when the nonuniform Fourier data has been interpolated to uniform Fourier coefficients.

7.1. Performance comparison. Several techniques extend traditional Fourier reconstruction methods to the nonuniform case for the purpose of generating MR images from nonuniform Fourier data, [8, 11, 20, 26]. Two popular techniques are convolutional gridding (with

Figure 17: Minmod enhancement, (14), using σ^G , σ^L , σ^T and σ^E , for Examples 2, (left), and 4, (right). Here we use log-spaced sampling, (24) with (23), with $N = 64$ and $N = 128$.

density compensation) and uniform resampling. Both represent methods that interpolate the nonuniform Fourier data to the uniform coefficients. One major practical consideration in applications involving non-Cartesian sampling of Fourier data is computational cost. These algorithms are popular since they allow the fast Fourier transform (FFT) to be employed. Unfortunately, as mentioned in the introduction, the methods inevitably suffer from interpolation errors and/or ill-conditioning, making any further processing inaccurate and difficult.

The convolutional gridding method modifies the partial Fourier sum, (4), for the nonharmonic coefficients, $\hat{f}(\lambda_n)$, with density compensation factors, α_n , in order to ‘compensate’ for the nonequispaced quadrature (in frequency space) of the inverse Fourier operator. A typical value for α_n is $\alpha_n = \lambda_{n+1} - \lambda_n$, which reflects the appropriate quadrature weights for the trapezoidal rule. The function is approximated by

$$(38) \quad f(x) \approx \sum_{n=-N}^N \alpha_n \hat{f}(\lambda_n) e^{i\lambda_n x}.$$

Using a convolution technique, the approximation can be manipulated so that the IFFT can be used.⁷

The uniform resampling technique uses the relationship

$$(39) \quad \hat{f}(\lambda_n) = \sum_{m=-\infty}^{\infty} \hat{f}(m) \text{sinc}(\lambda_n - m)$$

to approximate uniformly spaced $\hat{f}(m)$, $m = -M, \dots, M$, from nonharmonic measurements. Specifically, (39) is truncated to construct the system of equations $b = Ay$, where $y = (\hat{f}(m))_{m=-M, \dots, M}$, $b = (\hat{f}(\lambda_n))_{n=-N, \dots, N}$, and $A = (a_{nm})_{n=-N, \dots, N}^{m=-M, \dots, M}$, where $a_{nm} = \text{sinc}(\lambda_n - m)$. More information on uniform resampling algorithms can be found in [21].

⁷The convolutional gridding method is so named because the density compensation factors can also be expressed as $\alpha_n = \frac{1}{(f * \hat{\varphi})(\lambda_n)}$, where $f = \sum_{n=-N}^N \hat{f}(\lambda) \delta(\lambda - \lambda_n)$ and φ is an interpolating function (a function with energy concentrated in as small a band as possible in both Fourier and physical space), [19].

Figure 18: Log plot of the interpolation error of uniform Fourier data given log-spaced data, (24), with (left) uniform resampling and (right) convolutional gridding for Example 2. Both plots use $N = 128$.

Because this system is ill-posed, the most common practice is to compute the least-squares, minimum norm solution. After this interpolation step is completed, the standard concentration factor method from Section 2 can be directly applied. However, both the convolutional gridding and uniform resampling techniques have difficulties in accurately recovering the high frequency coefficients. Figure 18 illustrates the error in the high frequency coefficients obtained by uniform resampling and convolutional gridding for Example 2 when log-spaced data (24) is originally given. Since the concentration factor method depends upon information from the high frequency coefficients, such interpolation schemes can adversely affect its ability to locate edges successfully.

Figures 19, 20 and 21 compare the concentration factor method performance on the resampled Fourier coefficients, [26], to our new frame-theoretic edge detection method. The results are post-processed by the *minmod* algorithm, (14), with (10)-(12) used in the resampling schemes and σ^G , σ^L , σ^T , and σ^E used in the frame-theoretic edge detection method. The figures demonstrate that the oscillations are more pronounced for the interpolated schemes. This may yield to edges being falsely identified in smooth regions. The frame-theoretic edge detection method clearly reduces these oscillations.

The most important feature of our method is that it does not require interpolation in Fourier space. This is especially valuable when available data is sparse, such as in the log-sampled case. As is demonstrated in our figures, the frame-theoretic edge detection method gives results comparable to the interpolated case when the sampling sequence is somewhat regular, such as in the jittered case. However, as the sampling pattern grows more erratic, the quality of the frame-theoretic jump approximation degrades much more slowly than that of the concentration factor method based on interpolated Fourier data. Hence our new edge detection method appears to be more robust to the sampling pattern.

It may be possible to truncate the concentration method so that it only uses Fourier coefficients that are not affected by interpolation error. Unfortunately, such a truncation may impact the method's ability to resolve the jump function, especially if the underlying piecewise-smooth function has a lot of variation. Essentially, we would be giving up high frequency information all together.

Figure 19: Edge detection for Example 2, with jittered sampling, (21) with (23), (left) and log-spaced sampling, (24), (right), each with $N = 64$. Each figure compares the frame based concentration factor method with the standard concentration factor method using convolutional gridding and uniform resampling. The *minmod* enhancement is used in all cases.

Figure 20: Edge detection for Example 3, with jittered sampling, (21) with (23), (left) and log-spaced sampling, (24), (right), each with $N = 64$. Each figure compares the frame based concentration factor method with the standard concentration factor method using convolutional gridding and uniform resampling. The *minmod* enhancement is used in all cases.

7.2. Cost comparison. Recall that the relative L^2 reconstruction error (or simply relative error) for an approximation f_k to a function f (given, say, by Theorem 3) is the ratio $\frac{\|f - f_k\|}{\|f\|}$. In this specific case, f will be the jump function of the given signal, and f_k will be the jump approximation after k iterations of the frame algorithm.

Let $\delta > 0$ be the desired error bound, and let A and B be the lower and upper frame bounds (16) of the frame in question. We first calculate the number of iterations, N_{alg} , necessary to guarantee that $\|f - f_k\|/\|f\| < \delta$ for all $k \geq N_{\text{alg}}$ based on the error δ and the ratio $\kappa = B/A$.

Figure 21: Edge detection for Example 4, with jittered sampling, (21) with (23), (left) and log-spaced sampling, (24), (right), each with $N = 64$. Each figure compares the frame based concentration factor method with the standard concentration factor method using convolutional gridding and uniform resampling. The *minmod* enhancement is used in all cases.

With the accelerated frame algorithm, we have a convergence rate of $\frac{\|f-f_k\|}{\|f\|} \leq \frac{2\rho^k}{1+\rho^{2k}}$, where $\rho = \frac{\sqrt{B}-\sqrt{A}}{\sqrt{B}+\sqrt{A}}$. As shown in [18], $\log \rho \approx \frac{2}{\sqrt{\kappa}-1}$ when κ is large, and hence in order to have $\rho^{N_{\text{alg}}} < \epsilon/2$, then we must have $N_{\text{alg}} > \frac{\log \frac{2}{\epsilon}}{\log \frac{1}{\rho}} \approx \frac{\sqrt{\kappa}-1}{2}(\log 2 + \log \frac{1}{\epsilon})$.

Furthermore, each iteration (after the first, when the inner products are given) of the frame algorithm requires the calculation of inner products when applying the frame operator. From the finite frame operator, (25), we have that computing $S_N(f - f_k)$ as defined in Theorem 3 costs $\mathcal{O}(NN_{\text{quad}})$ operations, where N_{quad} is the number of quadrature points used to numerically integrate $\langle f - f_k, e^{i\lambda_n x} \rangle$. Here we use the trapezoidal rule. Although increasing N_{quad} would result in a more accurate approximation, setting $N_{\text{quad}} = \mathcal{O}(N)$ is adequate since all of the functions in question are piecewise smooth. Therefore, the total cost to compute the jump function is $\mathcal{O}(N_{\text{alg}}NN_{\text{quad}})$.

Alternately, for a fixed sampling scheme, the dual frame $\{S_N^{-1}e^{i\lambda_n x}\}_{n=-N}^N$ can be computed off-line and then stored, albeit at the higher cost of $\mathcal{O}(N^2N_{\text{alg}}N_{\text{quad}})$. After the dual frame is computed, the modified finite frame approximation, (30), represents a cost of $\mathcal{O}(N^2)$, since we can use (30) instead of the frame algorithm to compute $[f]$. A brief summary of the cost of other edge detection methods is given in Table 22. The computational cost of the other methods presented in this table is discussed in-depth in [27].

8. REMARKS

This paper presents an alternative method for computing jump locations given nonuniform Fourier data. The method is based on the concentration factor edge detection method, originally derived in [15] for uniform Fourier data. The method does not require interpolation in Fourier space, however. Rather it applies elements from frame theory to directly use the nonuniform Fourier data. In some cases, in fact, the resulting coefficients do not even constitute a frame. Nevertheless, our new method provides structure for determining edges

Method	Cost	Notes
Convolutional grid- ding, then (7)	$\mathcal{O}(N \log N + CN)$	N - number of coefficients. C - convolution width, usually $C \ll N$.
Uniform resampling, then (7)	$\mathcal{O}(CN \log CN + CM)$	N - number of coefficients. C - oversampling factor. M - small multiple of N .
Frame-theoretic edge detection, (30)	$\mathcal{O}(N^2)$	N - number of coefficients. Note that this cost estimate assumes that the dual frame has been precomputed.

Figure 22: Comparative costs of various spectral edge detection algorithms.

as though we are given frame coefficients. The benefits of using our method is that no interpolation error or ill-conditioning effects have impact on determining the edges. This is especially valuable in cases where there is sparse sampling in the high frequency modes, since maintaining accurate information there is crucial in determining edges. Non-Cartesian MR sampling trajectories are prime examples of these situations, since the spiral sampling yields dense sampling in the low modes and sparseness in the high modes. When interpolation is done in such cases, the resulting high frequency modes become increasingly inaccurate, [27].

Future investigations will include a more comprehensive study of sampling sequences for which using our frame exponential set up provides a robust and accurate technique for recovering jump discontinuities. We will also incorporate other artifacts into the data, such as noise and motion blur.

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E-mail address: `ag@math.la.asu.edu`

E-mail address: `taylor.hines@asu.edu`

SCHOOL OF MATHEMATICAL AND STATISTICAL SCIENCES, ARIZONA STATE UNIVERSITY, P.O. BOX 871804, TEMPE, AZ 85287-1804

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