Theorem 1 *Let $S \subset R^n$ and define H to be the set of all points $z \in R^n$ such that for some positive integer k, some points $z_1, \dots z_k \in S$, and some positive numbers $t_1, \dots t_k$ with $\sum_{i=1}^k t_i = 1$*

$$z = \sum_{i=1}^{k} t_i z_i.$$

Then H is a convex hull of S.

Proof. It is enough to show the following three facts.

1. H contains S
2. H is convex
3. Every convex set containing S must contain H

Part (1) is clearly true. Simply take $k = 1$ and $t_1 = 1$ to see that all points from S are special cases of formula for z.
Part (2) requires more work. Let $x, y \in H$ and let $z = tx + (1-t)y$. Since $x \in H$ and $y \in H$ there are points $x_1, \dots x_r$ and $y_1, \dots y_s$ all in S and positive numbers $p_1, \dots, p_r, q_1, \dots, q_s$ such that

$$x = \sum_{i=1}^{r} p_i x_i, \sum_{i=1}^{r} p_i = 1 \tag{1}$$

and

$$y = \sum_{i=1}^{s} q_i y_i, \sum_{i=1}^{s} q_i = 1. \tag{2}$$

Let

$$z_i = x_i, i = 1 \dots r,$$
$$z_{r+i} = y_i, i = 1 \dots s$$

and

$$t_i = tp_i, i = 1 \dots r,$$
$$t_{r+i} = (1-t)q_i, i = 1 \dots s.$$

Then

$$\sum_{i=1}^{r+s} t_i z_i = \sum_{i=1}^{r} t_i z_i + \sum_{i=r+1}^{s} t_i z_i =$$
$$= t \sum_{i=1}^{r} p_i x_i + (1-t) \sum_{i=1}^{s} q_i y_i = tx + (1-t)y = z.$$

Also

$$\sum_{i=1}^{r+s} t_i = t \sum_{i=1}^{r} p_i + (1-t) \sum_{i=1}^{s} q_i = t + (1-t) = 1.$$

Thus z is a convex combination of points from S and so $z \in H$.

Now let us prove part (3). We shall do this by induction on k in the defitnion of z.

- (*base case*) Case $k = 1$ is clear as points from S by definition are in C.
- (*inductive step*) Assume all points z with upper index in summation $k-1$ are in C and consdier $z = \sum_{i=1}^{k} t_i z_i$ with $\sum_{i=1}^{k} t_i = 1$. Define $t = 1 - t_k$ and let

$$r_i = \frac{t_i}{t}, i = 1, \ldots k-1.$$

We have $0 < t < 1$ and

$$\sum_{i=1}^{k-1} r_i = \frac{1}{t} \sum_{i=1}^{k-1} t_i = \frac{1}{t}(1 - t_k) = 1.$$

Thus, by inductive assumption,

$$z' = \sum_{i=1}^{k-1} r_i z_i \in C.$$

Since $z_k \in S$, z_k is also in C and C is a convex set thus it contains

$$tz' + (1-t)z_k$$

but

$$tz' + (1-t)z_k = t \sum_{i=1}^{k-1} r_i z_i + t_k z_k = \sum_{i=1}^{k} t_i z_i = z.$$

Therefore z is in C.