Linear Programming

Lecture 22

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Covers and matchings

A *graph* is a network whose arc have no directions.
A set of nodes is called a *cover* if it includes at least one endpoint of each arc.
A set of arcs is called a *matching* if no two of its arcs share a common endpoint.

Fact 1 *Let C be a cover and M be a matching. Then*

$$|M| \leq |C|.$$

Bipartite graphs

A graph is called bipartite if its nodes can be labeled "left" and "right" in such a way that each arc has one endpoint among the left nodes and the other among the right nodes.

Examples:

- Every tree is a bipartite graph.
- Hypercube $Q = (V, A)$ where V is the set of all binary strings of length n. Two strings are connected if and only if they differ on exactly one position.

Network Approach

Consider a bipartite graph with l vertices in the left part and r vertices in the right part. Construct network from the graph as follows:

- add unit supply to each left node and unit demand to each right node
- direct arcs from left to right
- add two new nodes v, w, arc wv, arcs iv for every left node i, wj for every right node j
- demand at v is l, supply at w is r.

Costs

Define costs: $c_{ij} = 0$ everywhere except $c_{wv} = -1$.
Then an integer-valued solution of
minimize cx subject to $Ax = b$, $x \geq 0$
maximizes x_{wv} and so gives the value of the largest matching.

Konig-Egervary

Theorem 2 *In every bipartite graph, the largest size of a matching equals the smallest size of a cover.*