Linear Programming

Lecture 16

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Game of Morra

Two players play the game. Each player hides one or two francs and guesses the amount of francs the other player has hidden. If only one player guesses correctly then the player receives the amount equal to the total of francs hidden. For example P1 hides 1 guesses 2, P2 hides 2 guesses 2; P1 wins 3 francs.

Notation: $[x, y]$ - hide x, guess y.

Strategies

Possible choices for players: $[1, 1]$, $[1, 2]$, $[2, 1]$, $[2, 2]$. These are called *pure strategies*.

Suppose P1 played: $[1, 1]$ in c_1 rounds, $[1, 2]$ in c_2 rounds, $[2, 1]$ in c_3 rounds, $[2, 2]$ in c_3 rounds.
P2 chooses between $[1, 2]$, $[2, 1]$ by flipping a fair coin.

Question: What is the average amount won by P2, when she uses the above strategy?

Answer: $\frac{c_1 - c_4}{2}$

Payoff matrix

Rows correspond to choices of a row player (RP) columns correspond to choices of a column player (CP); $a_i j$ is the payoff to RP when row i and column j are selected.

Let x_i be the frequency RP selects the ith row, y_j the frequency CP selects the jth column. The average payoff:

$$\sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij} x_i y_j .$$

Note that $\sum x_i = 1$, $\sum y_j = 1$, $x_i, y_j \geq 0$ (*stochastic vectors*).

Payoff

Let $x = [x_1 \ldots x_m], y = [y_1, \ldots y_n]^T$. Then the payoff can be written as

$$xAy$$

and if RP uses mixed strategy x he is guaranteed to win at least

$$min_y xAy$$

Of course, a normal RP wants to maximize this amount by selecting a suitable strategy x. Strategy x that maximizes $min_y xAy$ is called *optimal*.

Properties

Fact 1

$$min_y xAy = min_j \sum_{i=1}^{m} a_{ij} x_i.$$

In other words the minimum on the left hand side is attained by a pure strategy y, i.e. vector y which is stochastic and has one on exactly one coordinate.

How to find an optimal strategy?

Thus we want to solve the following problem:

maximize $min_j \sum_{i=1}^{m} a_{ij} x_i$
subject to: $\sum_{i=1}^{m} x_i = 1$
$x_i \geq 0, i = 1, \ldots, m$

which doesn't look to friendly but fortunately is equivalent to **maximize** z
subject to: $z - \sum_{i=1}^{m} a_{ij} x_i \leq 0, j = 1, \ldots, n$
$\sum_{i=1}^{m} x_i = 1$
$x_i \geq 0, i = 1, \ldots, m$

which you can (and want to) solve using the revised simplex method.