Linear Programming

Lecture 14

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General LP problems

maximize $\sum_{j=1}^{n} c_j x_j$

subject to:

$$\sum_{j=1}^{n} a_{ij} x_j \leq b_i, \quad (i \in I)$$
$$\sum_{j=1}^{n} a_{ij} x_j = b_i, \quad (i \in E)$$
$$x_j \geq 0\,, \quad (j \in R)$$

I-inequalities, E-equalities, R-variables that are explicitly restricted.
If j is not in R then x_j is called *free*. F - set of j's such that x_j is free.

The dual problem

minimize $\sum_{i=1}^{m} b_i y_i$
subject to:

$$\sum_{i=1}^{m} a_{ij} y_i \geq c_j, \quad (j \in R)$$
$$\sum_{i=1}^{m} a_{ij} y_i = c_j, \quad (j \in F)$$
$$y_i \geq 0\,, \quad (i \in I)$$

Example

Find the dual of the following problem:
maximize $3x_1 + 2x_2 + 4x_3$
subject to:

$$x_1 + 4x_2 + 2x_3 = 3$$
$$2x_1 + 2x_2 + x_3 = 1$$
$$x_1 - x_2 + 2x_3 \geq 4$$
$$x_1 \leq 2, 0 \leq x_2, 0 \leq x_3 \leq 1$$

Example cont.

First write the problem in an equivalent form:
maximize $3x_1 + 2x_2 + 4x_3$
subject to:

$$x_1 + 4x_2 + 2x_3 = 3$$
$$2x_1 + 2x_2 + x_3 = 1$$
$$-x_1 + x_2 - 2x_3 \leq -4$$
$$x_1 \leq 2$$
$$x_3 \leq 1$$
$$x_2 \geq 0, x_3 \geq 0$$

Example cont.

$R = \{2, 3\}$, $F = \{1\}$ and so the dual is

minimize $3y_1 + y_2 - 4y_3 + 2y_4 + y_5$
subject to:

$$y_1 + 2y_2 - y_3 + y_4 = 3$$
$$4y_1 + 2y_2 + y_3 \geq 2$$
$$2y_1 + y_2 - 2y_3 + y_5 \geq 4$$
$$y_3, y_4, y_5 \geq 0$$

Weak duality theorem

Fact 1 *For any solution* $\bar{x}_1, \ldots, \bar{x}_n$ *to the primal and any solution* $\bar{y}_1, \ldots, \bar{y}_m$ *to the dual we have*

$$\sum_{j=1}^{n} c_j \bar{x}_j \leq \sum_{i=1}^{m} b_i \bar{y}_i .$$

Duality theorem

Theorem 2 *If a linear programming problem has an optimal solution, then its dual has an optimal solution and the optimal values of two problems coinside.*