

## Irrationality of $\sqrt{2}$ , $\sqrt{3}$ , $\sqrt{5}$ , and $\sqrt{7}$

**Fact 1** For every integer  $n$ ,  $n$  is even if and only if  $n^2$  is even.

**Fact 2**  $\sqrt{2}$  is irrational.

**Proof.** Assume by the way of contradiction that  $\sqrt{2} = \frac{p}{q}$  where  $p, q$  have no common factors. Then

$$2q^2 = p^2$$

and since  $p$  is an integer, from Fact 1, we have  $p$  even. Therefore,  $p = 2k$  for some integer  $k$ . Then

$$q^2 = 2p^2$$

and so  $q^2$  is even. Since  $q$  is an integer, by Fact 1,  $q$  even. This is a contradiction as both  $p, q$  are even and  $p, q$  have no common factors.  $\square$

**Fact 3**  $\sqrt{3}$  is irrational.

**Proof.** Suppose  $\sqrt{3} = \frac{p}{q}$  where  $p, q$  are positive integers and  $q$  is the smallest possible. Note that  $1 < \frac{p}{q} < 2$  and so  $p < 2q$  and  $q < p$ . Then

$$\frac{3q - p}{p - q} = \frac{3 - \sqrt{3}}{\sqrt{3} - 1} = \sqrt{3}$$

but both  $3q - p, p - q$  are positive integers and  $p - q < q$ . This is a contradiction as the denominator was the smallest.  $\square$

**Fact 4**  $\sqrt{5}$  is irrational.

**Proof.** For a positive integer  $i$ , let  $m_i = (\sqrt{5} - 2)^i$ . Then  $m_i = a_i + b_i\sqrt{5}$  where  $a_i, b_i$  are integers. If  $\sqrt{5} = \frac{p}{q}$  where  $p, q$  are positive integers then

$$m_i = a_i + b_i\sqrt{5} = \frac{qa_i + pb_i}{q}$$

and the numerator is an integer. Since  $m_i > 0$ , the numerator is at least one. Therefore,

$$\frac{1}{q} \leq m_i.$$

On the other hand  $(\sqrt{5} - 2) < 0.5$  as  $2.5^2 > 5$  and so  $m_i < 1/2^i$ . Thus

$$\frac{1}{q} < \frac{1}{2^i}$$

for every positive integer  $i$ . However the right hand side of the above inequality goes to zero as  $i$  goes to infinity.  $\square$

**Theorem 5 (Fundamental Theorem of Arithmetic)** *Every positive integer  $n \geq 2$  can be written uniquely as a product of prime numbers.*

**Fact 6**  $\sqrt{7}$  is irrational.

**Proof.** Suppose  $\sqrt{7} = \frac{p}{q}$ . Then

$$7q^2 = p^2. \tag{1}$$

Note that  $p$  cannot be one as  $\sqrt{7} > 1$  and  $q$  cannot be one either as  $\sqrt{7}$  is not an integer. If 7 appears  $k$  times in the prime factorizations of  $p$  then it appears  $2k$  times in the prime factorization of  $p^2$ . The same for  $q^2$ . Consequently 7 appears an odd number of times in the prime factorization of  $p^2$  by the left hand side of (1) and an even number of times based on the right hand side of (1). However, the prime factorization is unique by **Theorem 5**.  $\square$