More advanced topics.

1. Row and Column spaces

   - Let A be m by n. Basis of $Row(A)$ contains non-zero rows of the echelon form of A.
   - Let A be m by n. Basis of $Col(A)$ contains columns of A which correspond to pivot columns of the echelon form of A.
   - We have
     $$dim(Col(A)) = dim(Row(A))$$
     which is called the rank of A ($rank(A)$) and
     $$rank(A) + dim(Null(A)) = n.$$

2. Orthogonality

   - Let V be a subspace of R^n. Then $V^\perp$ contains all vectors u which are orthogonal to every vector from V.
   - Small facts about orthogonality:
     - If $v_1, \ldots, v_k$ are mutually orthogonal then they are linearly independent.
     - $(V^\perp)^\perp = V$.
     - The only vector in both V and $V^\perp$ is 0.
     - If S is such that $V = span(S)$ then u is in $V^\perp$ if and only if u is orthogonal to every vertex from S.
   - We have
     $$Row(A) = Null(A)^\perp.$$
     and as a corollary
     $$dim(V) + dim(V^\perp) = n$$
     for every subspace V of R^n.

3. Normal systems and least squares

- If $Ax = b$ is inconsistent we can try to solve an approximate system that is we can project b on $Col(A)$ to find vector p in the solution space which is closest to b.
- Normal system: $A^T A\bar{x} = A^T b$. Least squares solution is $\bar{x}$ and $p := A\bar{x}$.
- If A is $m \times n$ of rank n then $A^T A$ is non-singular.

4. Gram-Schmidt algorithm

- Start with $v_1, \ldots v_k$ which are linearly independent.
- Set $u_1 := v_1$ and then iterate and compute

$$u_{i+1} := v_{i+1} - \left(\frac{v_{i+1} \cdot u_1}{u_1 \cdot u_1} u_1 + \cdots + \frac{v_{i+1} \cdot u_i}{u_i \cdot u_i} \right).$$

- If $v_1, \ldots, v_k$ are mutually orthogonal then projection of b on $V := span(v_1, \ldots, v_k)$ is

$$p := \frac{b \cdot v_1}{v_1 \cdot v_1} + \cdots + \frac{p \cdot v_k}{v_k \cdot v_k}.$$

5. Eigenvalues and eigenvectors

- Eigenvalue λ, eigenvector v: $Av = \lambda v$ for nonzero v.
- Eigenvalues of A are the roots of $det(A - xI) = 0$ (characteristic equation).
- To find eigenvector associated with λ solve $(A - \lambda I)x = 0$. Eigenvectors associated with λ form a subspace called the eigenspace.

6. Diagonalization

- Matrices A and B are similar if $B = P^{-1}AP$.
- An $n \times n$ matrix A is similar to diagonal matrix D if and only if A has n linearly independent eigenvectors.
- (*) Eigenvectors associated with different eigenvalues are linearly independent.
- (*) If an $n \times n$ matrix A has n distinct eigenvalues that it is similar to a diagonal matrix.

(*) Not covered in class.