Chapter 6

Eigenvalues

Problems:

- Find A^{267}.

- Find e^A.

Observations:

- If A is a a diagonal matrix then $A^m = B$ where $B = [b_{ij}]$ such that $b_{ij} = a_{ij}^m$.

- If A is similar to a diagonal matrix D, i.e. $A = S^{-1}DS$, then

$$A^m = (S^{-1}DS)(S^{-1}DS)\cdots(S^{-1}DS) = S^{-1}D^mS.$$

Definition 1 *An $n \times n$ matrix is called diagonalizable if there exists a nonsingular matrix X such that $X^{-1}AX$ is a diagonal matrix.*

If A is diagonalizable and D is the diagonal matrix then

$$D = X^{-1}AX, A = XDX^{-1}.$$

Problem 1: Method for diagonalizing a matrix A.

Problem 2: Which matrices can be diagonalized?

Eigenvalues

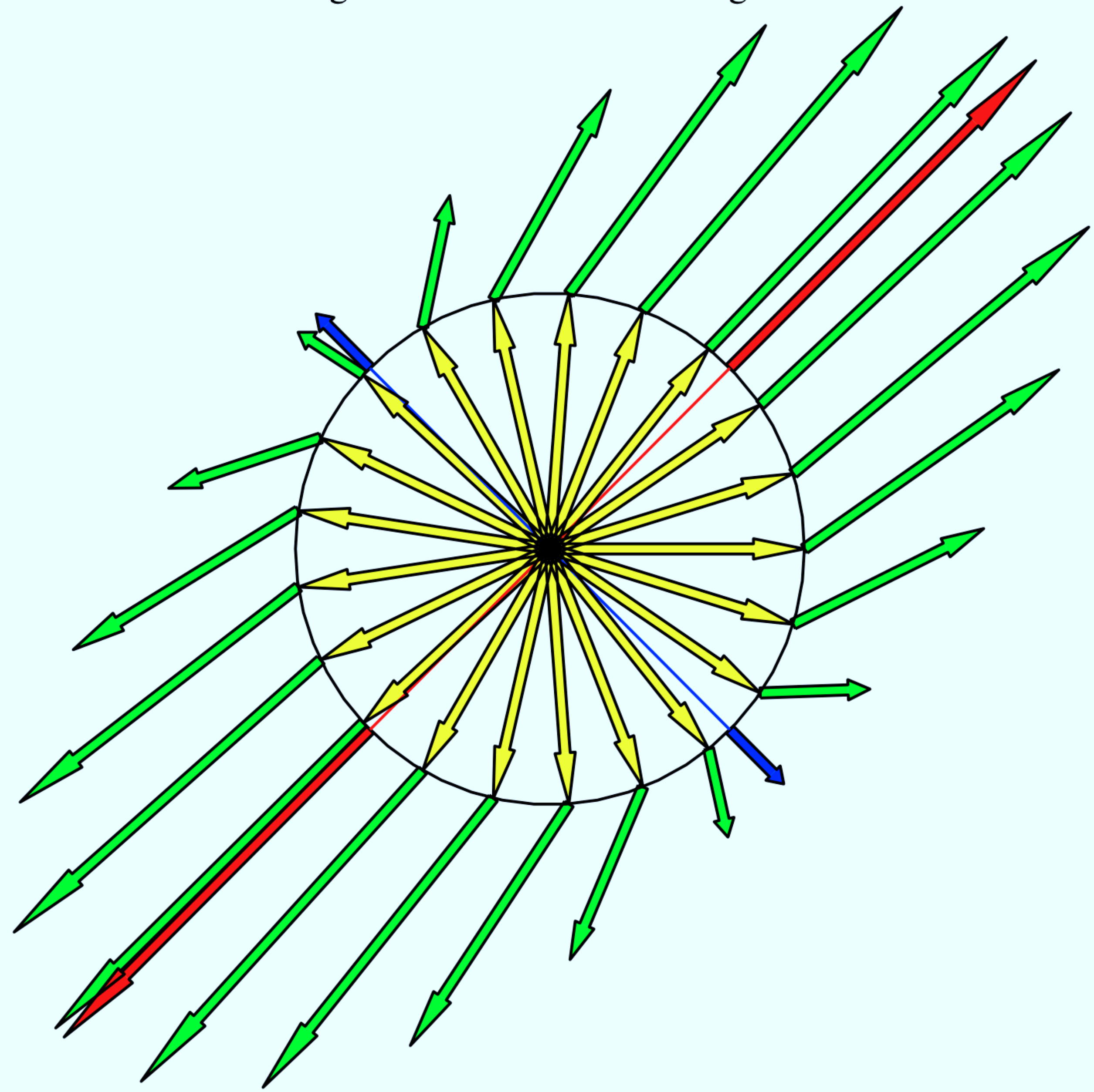
Definition 2 *Let A be an $n \times n$ matrix. A scalar λ is called an eigenvalue of A if there exists a nonzero vector v such that*

$$Av = \lambda v.$$

The vector v is called the eigenvector of A belonging to λ.

Fact 1 *Let A be an $n \times n$ and let λ be an eigenvalue of A. The set of all eigenvectors belonging to λ forms a subspace.*

The Images of Unit Vectors and Eigenvectors



Theorem 2 *The following conditions are equivalent.*

- λ *is an eigenvalue of* A.
- $(A - \lambda I)x = 0$ *has a nontrivial solution.*
- $N(A - \lambda I) \neq \{0\}$.
- $A - \lambda I$ *is singular.*
- $det(A - \lambda I) = 0$.

To find eigenvalues solve $p(\lambda) = 0$ where $p(\lambda) = det(A - \lambda I)$. The polynomial $p(\lambda)$ is called the characteristic polynomial of A and the roots of it are the eigenvalues.

Properties of eigenvalues.

Theorem 3 *Let A be $n \times n$ and let $\lambda_1, \ldots, \lambda_n$ be eigenvalues of A. Then*

- $det(A) = \lambda_1 \cdots \lambda_n$.
- $\sum_{i=1}^{n} a_{ii} = tr(A) = \sum_{i=1}^{n} \lambda_i$.

- *If B is similar to A then $\lambda_1, \ldots, \lambda_n$ are the eigenvalues of B.*

Diagonalization.

Theorem 4 *Nonzero eigenvectors that belong to distinct eigenvalues are linearly independent.*

Theorem 5 *An $n \times n$ matrix A is diagonalizable if and only if A has n linearly independent eigenvectors.*

Example 1 *Find A^{102} if*

$$A = \begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix}$$

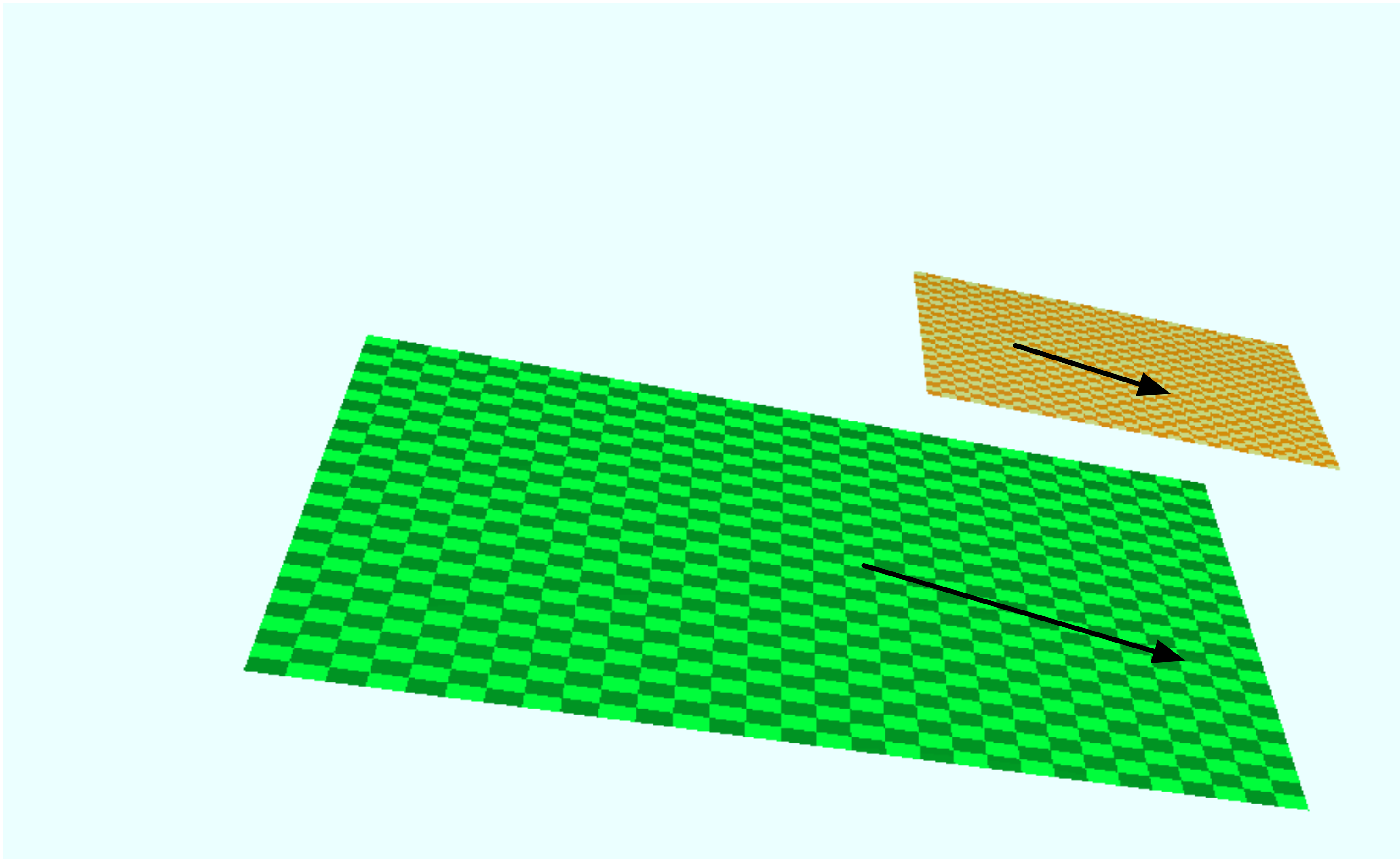
Example 2 *Find a matrix that diagonalizes A or state that no such matrix exists.*

(a)

$$A = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 4 & 0 \\ 1 & 0 & 2 \end{pmatrix}$$

(b)

$$A = \begin{pmatrix} 2 & 0 & 0 \\ -1 & 4 & 0 \\ -3 & 6 & 2 \end{pmatrix}$$



The eigenspace of A from (b) belonging to 2

Spectral Theorem

Theorem 6 *If A is a real symmetric matrix then there exists an orthogonal matrix U such that $U^{-1}AU$ is diagonal.*