Uniformity of sub-hypergraphs

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Graphs

- *Density:* The density of a bipartite graph $G = (V_1, V_2, E)$ is

$$d(V_1, V_2) = \frac{|E|}{|V_1||V_2|}.$$

- *ϵ-regular graphs* A bipartite graph $G = (V_1, V_2, E)$ is called (ϵ, d)-regular if for every $V_1' \subseteq V_1$ with $|V_1'| \geq \epsilon|V_1|$ and every $V_2' \subseteq V_2$ with $|V_2'| \geq \epsilon|V_2|$,

$$|d(V_1', V_2') - d| < \epsilon.$$

Example: Random graph $G(V, U, p)$ with fixed $0 < p < 1$ is (ϵ, p)-regular.

Slicing Lemma: Let $G = (V_1, V_2, E)$ be (ϵ, d)-regular and let $0 < \alpha < 1$. If $U_1 \subseteq V_1$ with $|U_1| \geq \alpha|V_1|$, $U_2 \subseteq V_1$ with $|U_2| \geq \alpha|V_2|$ then $G' = (U_1', U_2', E(U_1', U_2'))$ is (ϵ', d')-regular where $\epsilon' = \max\{\epsilon/\alpha, 2\epsilon\}$, $|d' - d| < \epsilon$.

Small sub-hypergraphs:[Duke,Rödl] For ϵ, d, α there exist ϵ', d' and K_0 such if $G = (U, V, E)$ is an (ϵ, d)-regular graph such that $n = |U| = |V|$ is large enough then for every $k \geq K_0$ all but $\alpha\binom{n}{k}^2$ graphs (U', V') with $U' \subseteq U$, $V' \subseteq V$, $|U'| = k = |V'|$ are (ϵ, d)-regular.

Algorithmic characterization: Let $G(U,V)$ be a bipartite graph.

- $G_1(\epsilon)$: all but ϵn vertices in U have degrees $(d \pm \epsilon)n$.
- $G_2(\epsilon)$: all but ϵn^2 pairs of vertices from U have co-degree $(d \pm \epsilon)^2 n$
- $G(\epsilon)$: (U,V) is (ϵ, d)-regular.

Lemma 1 *For every ϵ there is δ such that if $G(\delta)$ then $G_1(\epsilon)$ and $G_2(\epsilon)$.*

Lemma 2 (Alon, Duke, Lefman, Rödl, Yuster) *For every ϵ there is δ such that if $G_1(\delta)$ and $G_2(\delta)$ then $G(\epsilon)$.*

Lemma 3 *Let* $0 < \zeta < 1$ *be a constant. Let* $H = (A, B, E)$ *be a bipartite graph with* $|A| \geq \frac{2}{\zeta}$*. Set* $d_H(A, B) = \rho$*. Let* D *be the collection of all pairs* $\{a, a'\}$ *of vertices of* A *for which*

(i) $\deg_H(a), \deg_H(a') > (\rho - \zeta)|B|$, *and*

(ii) $|N_H(a) \cap N_H(a')| < (\rho + \zeta)^2 |B|$.

Then, if $|D| > \frac{1}{2}(1 - 5\zeta)|A|^2$*, graph* (A, B) *is* ζ'*-regular, where* $\zeta' = (16\zeta)^{\frac{1}{5}}$*.*

Note:

- The number of C_4's in a (ϵ, d)-regular graph (U, V) with $|U| = |V| = n$ is around $\binom{n}{2}\binom{d^2 n}{2} \approx d^4 n^4/4$.
- This in fact is equivalent to regularity.

How to use it to prove a small subgraphs fact?

Will assume for simplicity that all vertices have degree dn all pair have co-degree d^2n.

- Select U', V' randomly from $\binom{U}{k}$, $\binom{V}{k}$.

- For u degree $deg(u, V')$ with

$$E(deg(u, V')) = |N(u)|\frac{\binom{n-1}{k-1}}{\binom{n}{k}} = dk.$$

- For a fixed vertices $u \neq u'$, $|(N(u') \cap N(u)) \cap V'|$ has a hypergeometric distribution with

$$E(|(N(u') \cap N(u)) \cap V'|) = d^2k.$$

- Appeal to Chernoff (and other tools) to show that degrees will be around dk, co-degrees around d^2k.

Applications in Testing [Goldreich, Goldwasser, Ron]

Idea: Let P be a property of being 3-colorable. We say that graph G is ϵ-far from P if after deletion of any $\epsilon|V(G)|^2$ edges of G the resulting graph is not 3-colorable. A tester is an algorithm which given ϵ and graph G can query if there is an edge between two vertices. The algorithm accepts G if G has property P and rejects G with probability at least $2/3$ if G is ϵ-far from P.

Goal is to have a constant $C(\epsilon)$ number of queries.

- Imagine graph $\bar{G}$ obtained from G after applying the regularity lemma, deleting the exceptional class, and edges of irregular or small density pairs.

- $|E(G)| - |E(\bar{G})| \leq \epsilon n^2$.

- Consider the cluster graph $C(\bar{G})$. It is easy to argue that $\chi(C(\bar{G})) = \chi(\bar{G})$.

- Imagine a random subgraph on a large but constant number of vertices is selected. It will form a fingerprint of $\bar{G}$ and so if regularity of ”small” subgraphs is preserved it will give a fingerprint of $C(\bar{G})$.

Note: Random methods give better query complexity (in graphs).

3-uniform hypergraphs

- *Density:* The density of $H = (V_1, V_2, V_3, E)$ is

$$d(V_1, V_2, V_3) = \frac{|E|}{|V_1||V_2||V_3|}.$$

- *ϵ-regular hypergraphs:* $H = (V_1, V_2, V_3, E)$ is called (ϵ, d)-regular if for every $V_i' \subseteq V_i$ with $|V_i'| \geq \epsilon |V_i|$

$$|d(V_1', V_2', V_3') - d| < \epsilon.$$

Note: This notion does not give a "counting" result.

Result:

Theorem 4 *For all* $\rho, \alpha, \epsilon > 0$*, there exists* $\zeta > 0$ *and integers* K_0 *and* N_0 *so that whenever is a* (ρ, ζ)*-regular 3-uniform hypergraph* (V_1, V_2, V_3) *with* $|V_i| = N \geq N_0$ *then for* $K_0 \leq k$*, all but*

$$\alpha \binom{N}{k}^3$$

3-tuples of sets $W_i \in \binom{V_i}{k}$*,* $1 \leq i \leq 3$*, span* (ρ, ϵ)*-regular hypergraph.*

Comments:

- Alon, Fernandez de la Vega, Kannan, and Karpinski with constant 1/40
- Independently proved by Mubayi and Rödl using a different technique.

Algorithmic hypergraph regularity

Theorem 5 (C., Rödl) *There is a* $O(n^5 \log^2 n)$ *algorithm that finds an* ϵ*-regular partition of a 3-uniform hypergraph.*

Comment: Method does not give an ”easy” characterization of ϵ-regular 3-tuples.

Different regularity

- Let P be a graph and let $K_3(P)$ denote the system of triangles of P, i.e.,

  $$K_3(P) = \Big\{\{x, y, z\} : \{x, y\},\ \{x, z\},\ \{y, z\} \in P\Big\}.$$

- We say graph P *underlies* 3-graph H if $E(H) \subseteq K_3(P)$.

- $d_H(P) = |E(H)|/|K_3(P)|$ is the *density* of H with respect to P (provided $|K_3(P)| > 0$).

Minimality

Assume:

1. H and P have common 3-partition $V = V(P) = V_1 \cup V_2 \cup V_3$, $|V_i| = N$, $1 \leq i \leq 3$.
2. P^{ij} is (d, ε)-regular for each $1 \leq i < j \leq 3$.
3. P underlies H and $d_H(P) = \alpha$.

Let $\left|K^{(3)}_{2,2,2}(H)\right|$ be the number of copies of the hypergraph $K^{(3)}_{2,2,2}$ in H.

Definition 1 ((α, δ)-minimal) *H is (α, δ)-minimal with respect to P if*

$$\left|K^{(3)}_{2,2,2}(H)\right| \leq \alpha^8 d^{12} \binom{n}{2}^3 (1+\delta)\,.$$

An *(l, t, ϵ)-partition* is a pair of partitions $\mathcal{V}$ and $\mathcal{B}$ satisfying the following properties:

1. $\mathcal{V}$ is a vertex partition given by

$$V_a = V_{a1} \cup \cdots \cup V_{at}, \quad |V_{aj}| \approx N/t.$$

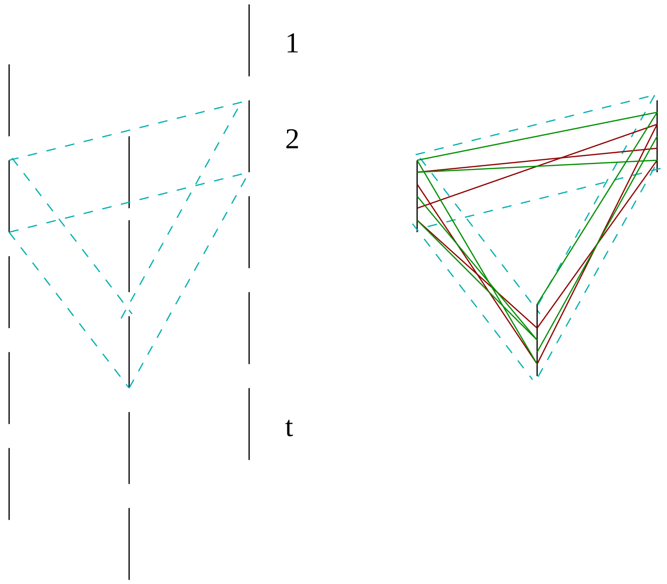
2. $\mathcal{B}$ is a partition of the complete graph $K^{(2)}(V_1, V_2, V_3)$ given by

$$K^{(2)}\left[V_{ai}, V_{bj}\right] = \bigcup_{1 \leq s \leq \ell} P_{ab}^{ij}(s)$$

where

$$P_{ab}^{ij}(s) \textit{ is } \left(l^{-1}, \epsilon\right)\textit{-regular.}$$

$\mathcal{P}$ is an (α_0, δ)-*minimal* partition if all but $\delta t^3 l^3$ triads P satisfying with $d_{H_P}(P) = \alpha_P \geq \alpha_0$ also satisfy that H_P is (α_P, δ)-minimal



Theorem 6 (Haxell, Nagle, Rödl) *For all* α_0 *and* $\delta > 0$ *and all functions* $\boldsymbol{\epsilon} : \mathbb{N} \to (0,1)$ *there exist integers* T_0*,* L_0 *and* N_0 *so that every 3-partite 3-graph* H *with 3-partition* $V = V_1 \cup V_2 \cup V_3$*,* $|V_a| = N > N_0$*,* $1 \leq a \leq 3$*, admits an* (α_0, δ)*-minimal* (l, t, ϵ)*-partition where* $\varepsilon = \boldsymbol{\epsilon}(l)$ *and* $l \leq L_0$ *and* $t \leq T_0$*.*

Comments:

- For general k the regularity was proved recently by Gowers.

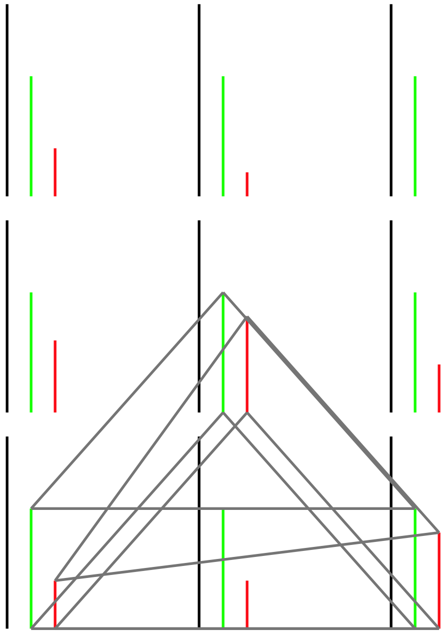
Main idea of the proof

1. Minimality part:

   - Prove an analog of the theorem for (H, P) in which the minimality is inherited.

   - Prove that if (H, P) is α-minimal then its density $d_H(P) \approx \alpha$.

2. Regularity part:



- Find an (α_0, δ)-minimal (l, t, ϵ)-partition and imagine $W_i \subset V_i$ with $|W_i| = k$ are selected at random.

- W_i will intersect most of the parts of V_{ia} in the right way i.e $\approx k/n|V_{ia}| \approx k/t$.

- Consider a subset $U_i \subset W_i$. Goal is to see $d(U_1, U_2, U_3) \approx d(W_1, W_2, W_3)$.

- Of course U_i's are not random and so their intersections with V_{ia} are not predictable but many intersections will be large.

- For simplicity assume that there is just one set in each partition. In addition assume all triads P are α_P-minimal (and so they preserve density). Then

  $$e(U_1, U_2, U_3) = \sum_P |E(U_1, U_2, U_3) \cap \Delta(P)|$$

  $$= \sum_P d_{H[U_1,U_2,U_3]}(P) |\Delta(P[U_1, U_2, U_3])|$$

  $$\approx \sum_P d_{H[W_1,W_2,W_3]}(P) \Delta(P[U_1, U_2, U_3])$$

  $$\approx |U_1||U_2||U_3| \frac{1}{l^3} \sum_P \alpha_P.$$

  In the same way

  $$e(W_1, W_2, W_3) \approx |W_1||W_2||W_3| \frac{1}{l^3} \sum_P \alpha_P.$$

Minimality part

Minimality can be characterized by the conditions on the co-degree in co-links assuming P_{ij}'s are d-regular (Nagle, Rödl).

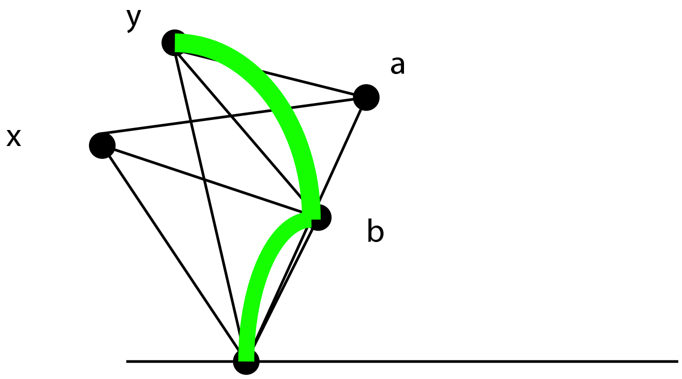
$$L_x = \{\{y,z\} | \{x,y,z\} \in H\}$$

$$L_{xy} = L_x \cap L_y.$$

- $H_1(\delta)$: (H,P) is (α,δ)-minimal.

- $H_2(\delta)$: All but at most δ fraction of cycles $\{x,y,a,b\}$ in P^{12} satisfy

$$deg_{L_{xy}}(a,b) = \alpha^4 d^4 n(1+\delta).$$



- $H_1(\delta)$ implies $H_2(\delta')$.

- $H_2(\delta)$ implies $H_1(\delta')$.