

Pebbling and Connectivity of Graphs

Andrzej Czygrinow
Department of Mathematics & Statistics
Arizona State University

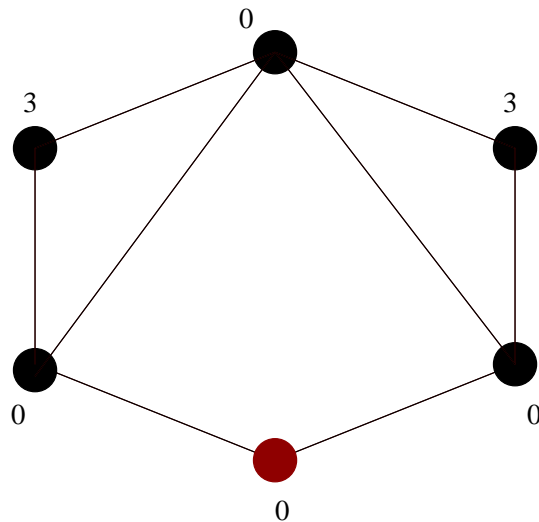
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Joint work with G. Hurlbert, H. Kierstead,
W. T. Trotter

Definitions

- *Pebbling function* on $G = (V, E)$ - function $f : V \rightarrow N_0$; $\|f\| = \sum_v f(v)$.
- *Pebbling step* - remove 2 pebbles from a vertex v and place 1 on a vertex adjacent to v .
- *Solvable function for r* - can place at least one pebble on r by a sequence of pebbling steps.
- *Solvable function* - solvable function for any vertex r .
- $\pi(G) = \min t$ such that any function f with $\|f\| = t$ is solvable in G .

Example:



$$\pi(G) \geq |V(G)|$$

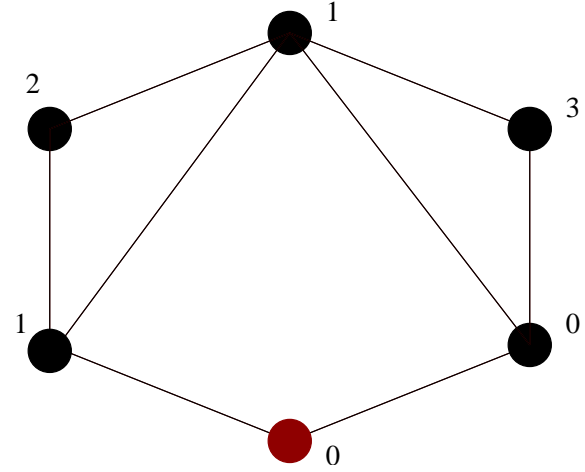
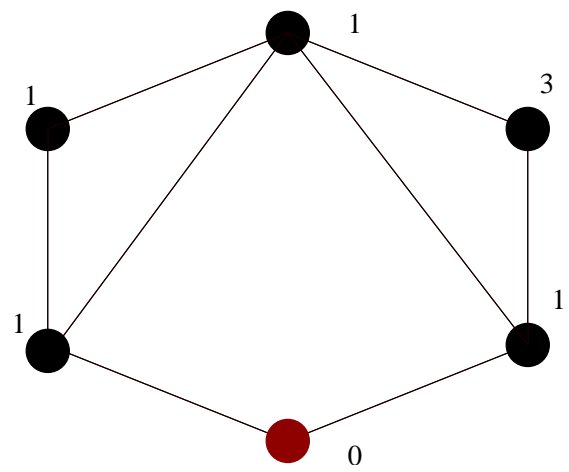
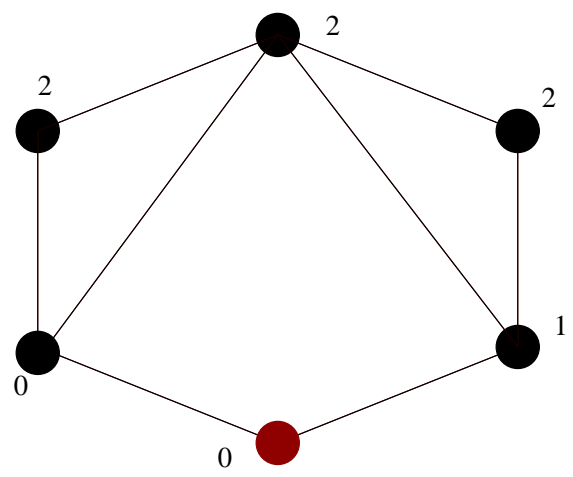
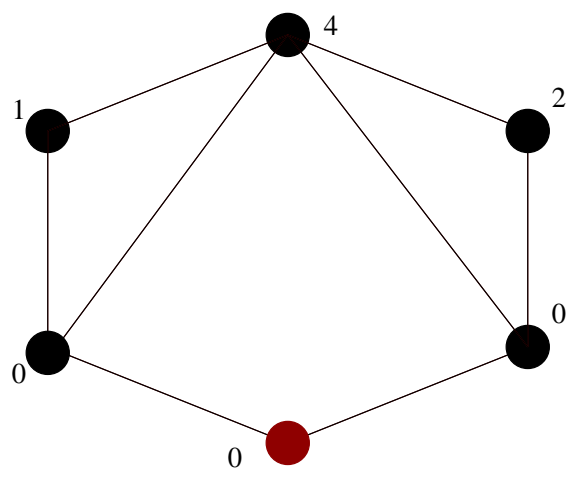
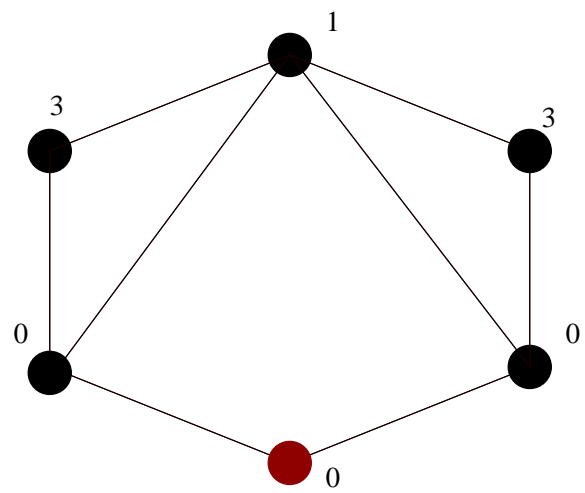
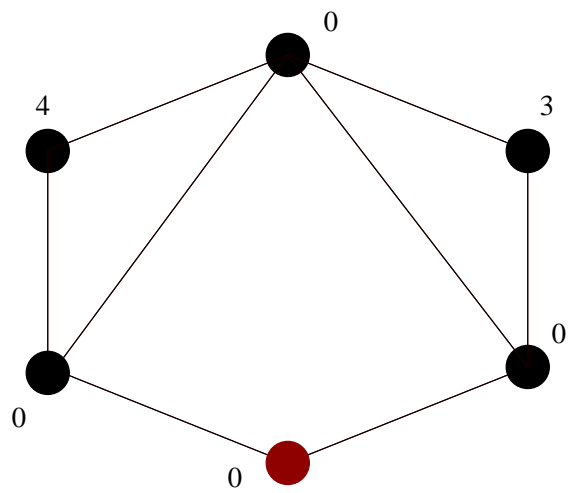
Question: For which G , $\pi(G) = |V(G)|$?

Example: $\pi(K_n) = n$, $\pi(K_{n,m}) = n + m$.

Theorem 1 (Chung, 89)

$$\pi(Q_n) = 2^n.$$

Goal: Find sufficient conditions for G to have $\pi(G) = |V(G)|$.



Conjecture 1 (Clarke, Hochberg, Hurlbert, 97) For every $d \geq 1$ there is the least integer $k = k(d)$ such that every graph G of diameter d and connectivity k has $\pi(G) = |V(G)|$.

- $d = 1$ is trivial.

- (Clarke, Hochberg, Hurlbert, 97)

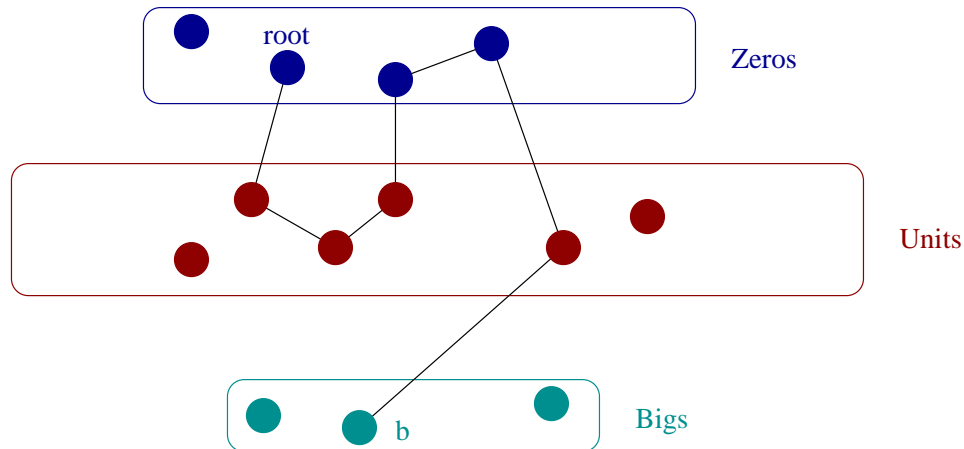
$$k(2) = 3.$$

- (Clarke, Hochberg, Hurlbert, 97)
Characterization of graphs G with $\text{diam}(G) = 2$ and $\pi(G) = |V(G)|$.
- $k(d) = \Omega(2^d/d)$ (blow-up of the path).

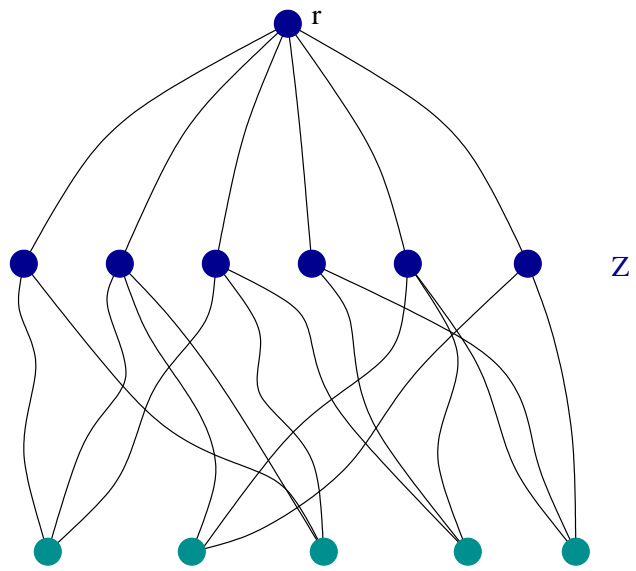
Theorem 2 (C., Hurlbert, Kierstead, Trotter)

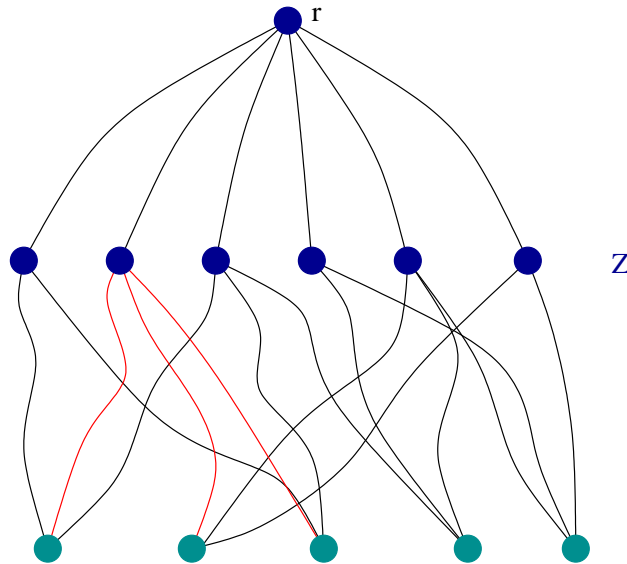
$$k(d) \leq 2^{2d+3}.$$

Idea of proof



- $\sum_{b \in B} f(b) = |B| + |Z|.$
- If $\omega = \sum_{b \in B} f(b) / |B|$ then $|Z| = |B|(\omega - 1) < |B|2^d$
- Find a large special set $B_0 \subseteq B$ and look at r and B_0 .





- If $B_0 = B$ then we would have $|Z| = |B|(\omega - 1)$, the number of paths starting in B is at least $k(d)|B| > 2^{2d}|B|$ so there is a vertex of degree larger than 2^d .
- However the paths from B do not need to be disjoint "below the blue set".

Two things must be modified for the argument to work

1. "Blue set" must be enlarged by the union of sets that separate each big vertex from other big vertices in a special subgraph.
2. B_0 must be selected with care so that b_j is not in the separation set of b_i whenever $i < j$.

Applications

1. Graham's conjecture

Conjecture 2 (Graham, 89) *For two connected graphs G and H ,*

$$\pi(G \times H) \leq \pi(G) \cdot \pi(H).$$

Partial results:

- Tree by Tree , Moews (1992).
- Cycle by cycle , Herscovici, Higgins (1998), Snevily, Foster (2000), Herscovici (2002).
- 2-pebbling property.

Theorem 3 (C., Hurlbert) Let G and H be connected graphs on n vertices with minimum degrees $d(G)$, $d(H)$ and let $d = \min\{d(G), d(H)\}$. If $d \geq 2^{12n/d+15}$ then

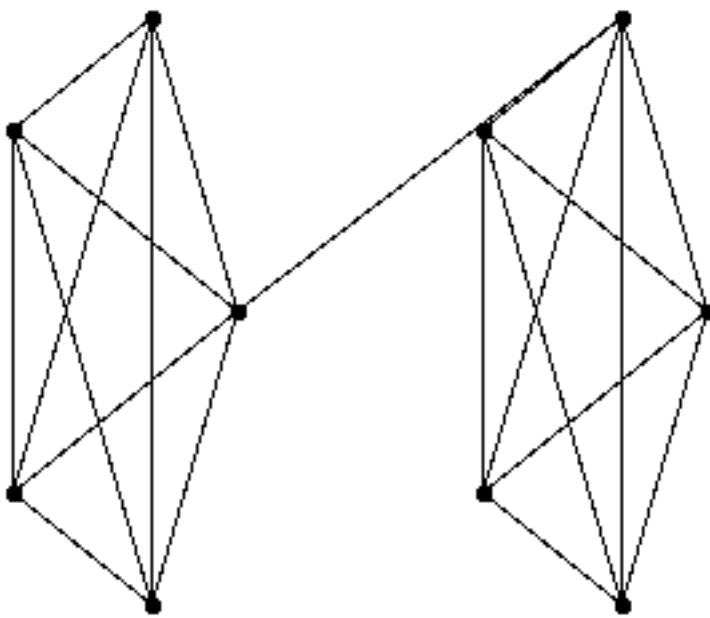
$$\pi(G \times H) \leq \pi(G) \cdot \pi(H).$$

Comments:

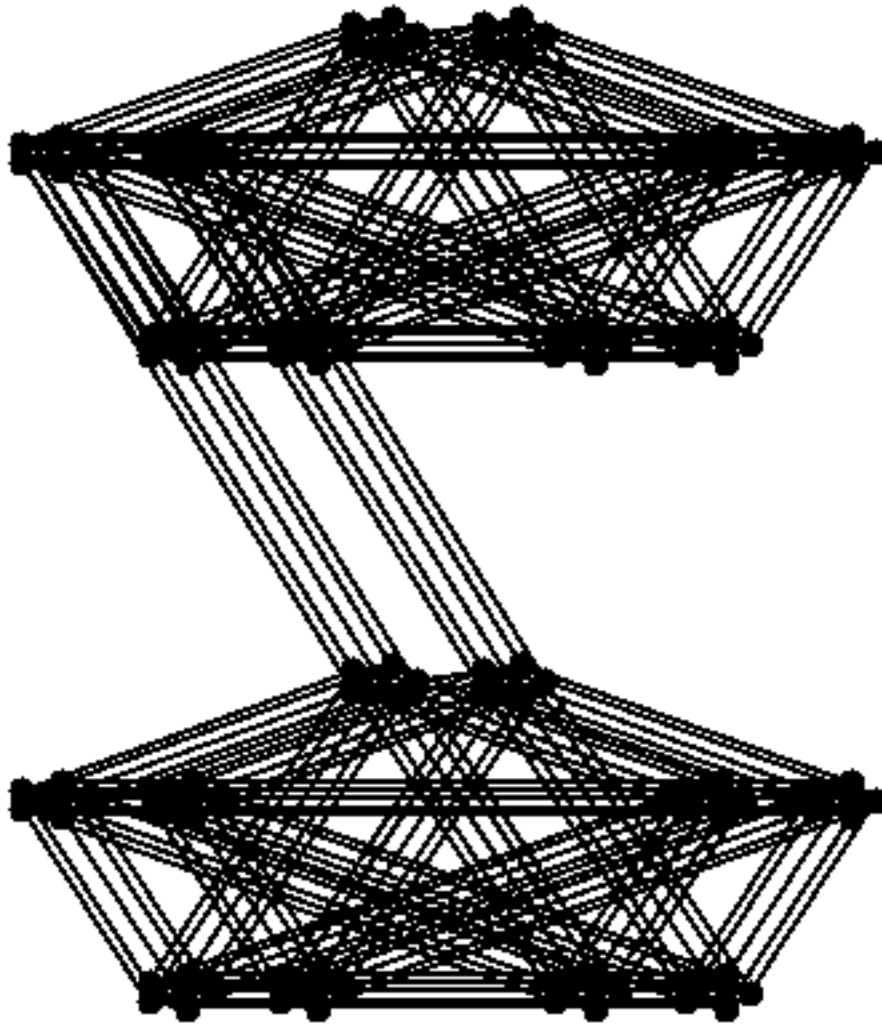
- $d \gg \frac{n}{\lg n}$.
- In fact $\pi(G \times H) = |V(G)||V(H)|$.

Outline of proof

- G and H are connected and their minimum degree is large $\rightarrow G$ and H have small diameter. Thus $G \times H$ has a small diameter.
- How about the connectivity of $G \times H$?
Observation: (C., Kierstead) If G and H are connected then $\kappa(G \times H) \geq \min\{\delta(G), \delta(H)\}$.
- $G \times H$ has small diameter and large connectivity so $\pi(G \times H) = |V(G)||V(H)|$.



G has connectivity one.



$\kappa(G \times G) = 8$ accordingly to Mathematica.

2. Girth and pebbling

Let $g_0(n)$ denote the maximum number g such that there exists a graph G on n vertices with $\text{girth}(G) \geq g$ and $\pi(G) = n$.

Easy:

$$g_0(n) = O(\lg n).$$

Theorem 4 (C. Hurlbert)

$$g_0(n) = \Omega(\sqrt{\lg n}).$$

Outline of proof

- Use the Erdős "construction": Consider $G(n, p)$ with $p = n^{-1+1/g}$. Expected number of cycles of lengths at most $g - 1$ is

$$\mu = \sum_{i=3}^{g-1} \binom{n}{i} \frac{(i-1)!}{2} p^i < gn^{1-1/g}$$

- Every degree is about pn (by Chernoff).
- Delete one edge from each "short" cycle to obtain girth at least g .
- The average degree of graph will still be large (about $n^{1/g}$).
- Apply Mader's Theorem which gives a $n^{1/g}/32$ -connected subgraph H .

- Add edges to H to have diameter at most $g - 1$ and girth g .
- Apply CHKT result about pebbling number of highly-connected graphs.

Conclusions

- Small diameter and large connectivity force a graph to have a small pebbling number.

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$$k(d) = \Omega(2^d/d),$$

$$k(d) = O(2^{2d}).$$