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# Distributed almost exact approximations for minor-closed families

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## Computational model

- Network is modeled as an undirected graph.
- Global clock. In a single round a vertex can send, receive, and compute.
- Local computations are unlimited.

## Goal:

- Deterministic algorithms.
- Time (the number of rounds) is poly-logarithmic in the order of a graph.

## Problems

- **Domination problems**

- The minimum dominating set problem (MD).
- The minimum connected dominating set problem (MCD).

- **Distributed almost-exact approximations**

Given a positive integer  $k$ , an algorithm finds in a poly-logarithmic number of rounds a solution with value of at most

$$(1 + O(1/\ln^k |G|))OPT.$$

## Minor-closed families of graphs

- $X$  is called a *minor* of  $Y$  if it can be obtained from a subgraph of  $Y$  by a sequence of edge contractions.

**Note:** Equivalent to contracting connected subgraphs.

- $\mathcal{C}$  is *minor-closed* if for every  $G \in \mathcal{C}$ , any minor of  $G$  is in  $\mathcal{C}$ .
- J. Nešetřil, P. Ossona de Mendez [2005, 2006] gave many equivalent conditions for a non-trivial  $\mathcal{C}$  to be minor-closed.
- We need:

$$\rho_{\mathcal{C}} = \sup_{G \in \mathcal{C}} \frac{\|G\|}{|G|}.$$

A non-trivial  $\mathcal{C}$  is minor-closed iff  $\rho_{\mathcal{C}} < \infty$ .

## Results

**Theorem 1** *Let  $\mathcal{C}$  be a proper minor-closed family of graphs. There is a distributed almost exact approximation which given  $q$  finds in a graph  $G \in \mathcal{C}$  a dominating set  $\bar{D}$  such that*

$$|\bar{D}| \leq \left(1 + o\left(\frac{1}{\ln^q |G|}\right)\right) \gamma(G)$$

*in  $O(\ln \ln |G| \ln^* |G| \ln^{1+r} |G|)$  rounds where  $r = 6(q + 1)\rho_{\mathcal{C}} \ln 3$ .*

**Theorem 2** *Let  $\mathcal{C}$  be a minor-closed family. There is a distributed almost exact approximation which given  $q$  finds in a connected graph  $G \in \mathcal{C}$  a connected dominating set  $\bar{D}$  such that*

$$|\bar{D}| \leq \left(1 + o\left(\frac{1}{\ln^q |G|}\right)\right) \gamma_{\mathcal{C}}(G)$$

*in  $O(\ln \ln |G| \ln^* |G| \ln^{1+r} |G|)$  rounds where  $r = 6(q + 1)\rho_{\mathcal{C}} \ln 3$ .*

## Related Work

- **Different models:** A lot ...
- **Randomized algorithms:** (1) L. Jia, R. Rajaraman, and R. Suel, (2) D. Dubhashi, A. Mei, A. Panconesi, J. Radhakrishnan, A. Srinivasan, (3) F. Kuhn, R. Wattenhofer
- **Different classes of graphs:** UDG (Kuhn et. al., ACMH)
- **Planar graphs:** Approximations for MaxIS (planar), MaxMatching (planar), MDS (outerplanar) (ACMH)

## General strategy of algorithms

- (1) Find a (special) clustering of  $G$ .
- (2) In each cluster find an optimal solution to the problem.
- (3) Return the union of these solutions.

## Properties of Clusters

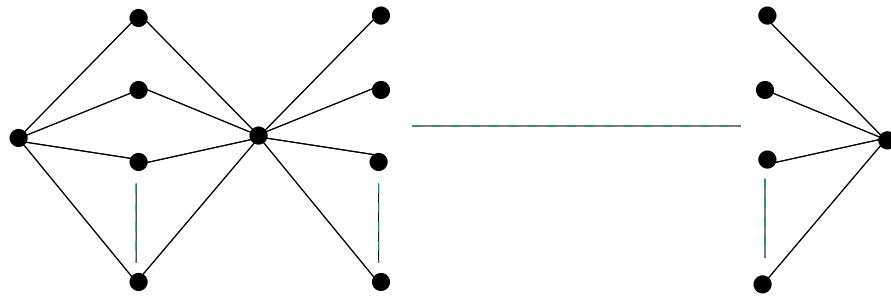
- Diameter of each cluster is poly-log in  $|G|$  (can compute locally).
- Error on the boundary of clusters is small (can disregard it). The number of edges connecting different clusters is small.

## We need more

Error with respect to what?

## Remarks

- Works for approximation of MaxIS
- Solution for MaxDS can be small with respect to the number of vertices



## What we need ...

...is a small error with respect to the value of an optimal solution.

## Preprocessing:

**Goal:** Find a  $O(\log |G|)$ -approximation of the *MaxDS*.

## Greedy Algorithm

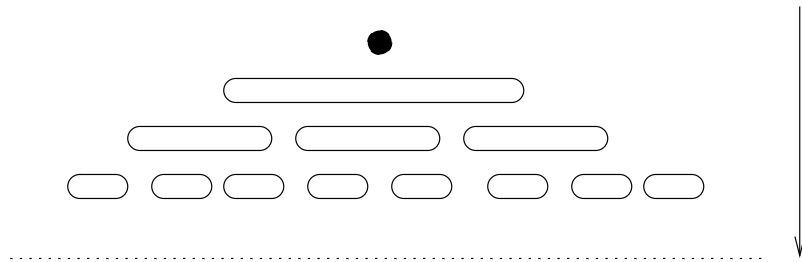
- Let  $V_1 := V$ ,  $V_2 := \emptyset$ .
- In the  $i$ th step add to the solution the set  $B$  of vertices  $v$  in  $G$  with  $|N(v) \cap V_1| \geq |G|/2^{i+1}$ . Move neighbors of  $B$  to  $V_2$  and delete  $B$  from  $G$ .
- Iterate  $O(\log |G|)$  times.

**Claim 3** *In each iteration set  $B$  satisfies*

$$|B| = O(\gamma(G)).$$

## Examples:

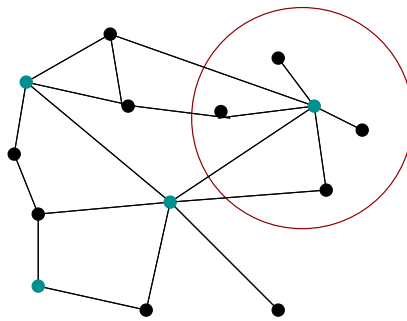
- If all vertices have constant degree then add **all of them**.
- **Intuition:** Delete  $B$ , keep  $V_2$  as possible vertices to be added to a dominating set, and compute degree to  $V_1$ .



- Proof needs bounded arboricity of  $G$ .

## Auxiliary graph

**Auxiliary graph:** Let  $D$  be a dominating set. Every vertex not in  $D$  joins one of the vertices from  $D$ .  $\mathbf{A}$  is the minor obtained by contracting these subgraphs.



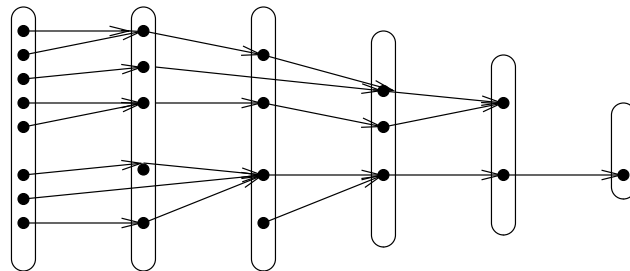
**Observation:** Auxiliary graph  $\mathbf{A}$  is in  $\mathcal{C}$  and

$$\gamma(G) \leq |\mathbf{A}| \leq O(\log |G| \gamma(G)).$$

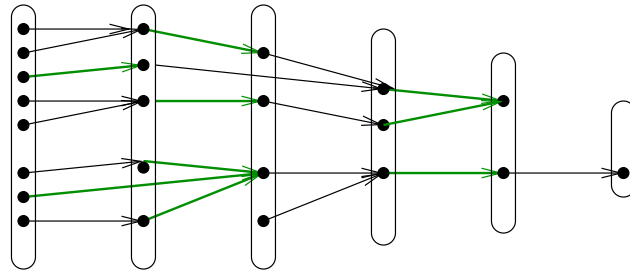
## Clustering procedure

We must consider weighted graphs with weights on edges.

- **Decompose** into  $O(\log |G|)$  independent sets (uses Cole-Vishkin). Each vertex has degree  $O(1)$  "to the right". Each vertex selects an edge of the largest weight.



- **Contract** subgraphs. Each component of  $H$  is a tree of diam  $O(\log |G|)$  and  $\omega(E(H)) = \Omega(\omega(E(G)))$ . Select heavy stars and contract. two.



- Iterate  $O(\log \log |G|)$  times.

## Analysis

Let  $n$  be given (such that  $|G| \leq n$ ).

**Lemma 4** *Let  $V_1, \dots, V_L$  be the clusters in  $G$  obtained from Clustering. Then we have:*

1. *For every  $i$ ,  $G[V_i]$  is the connected subgraph of diameter  $O(\ln^d n)$ , where*

$$d = 6c\rho_C \ln 3.$$

2. *The number of edges connecting different clusters is  $O(\|G\| / \ln^c n)$ .*
3. *Clustering runs in  $O(\ln \ln n \ln^* n \ln^{1+d} n)$  rounds.*

## Dominating set

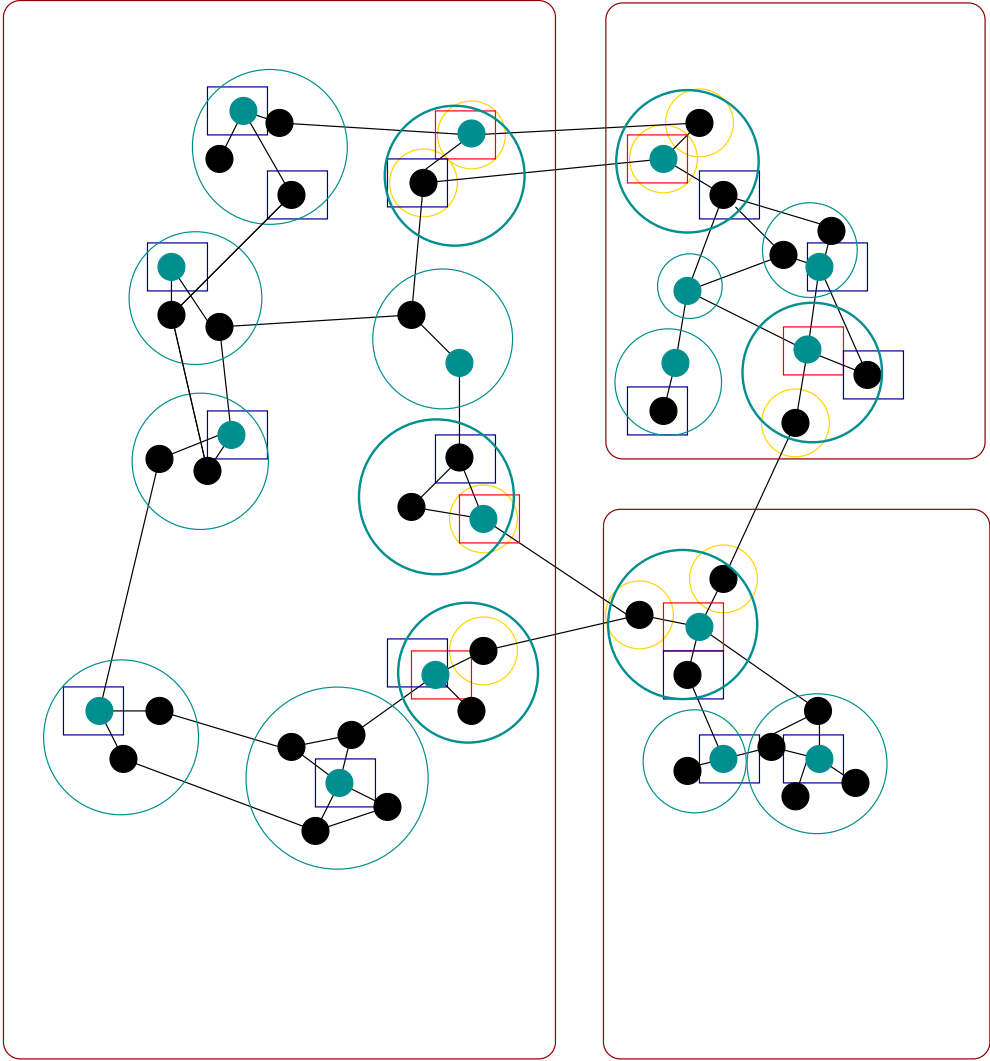
1. Find  $O(\log |G|)$ -approximation  $D_a$  and consider  $\mathbf{A}$ .
2. Use Clustering to cluster  $\mathbf{A}$  ( $n = |G|$ ).
3. In small clusters which are on the border:  
Add the **centers** (i.e. a vertex from  $D_a$ ) to our solution.
4. In each big cluster: Find an optimal solution  $D_i$  that dominates the interior.
5. Return  $\mathbf{D}$  which is the union of  $D_i$ 's and the centers.

## Why does it work?

- $|\mathbf{A}| = O(\log |G| \gamma(G))$  and the number of border clusters is  $O(|\mathbf{A}| / \ln^c |G|)$ .
- The number of **centers** is  $O(\gamma(G) / \ln^{c-1} |G|)$ .
- If  $D^*$  is optimal then  $|D^* \cap V_i| \geq |D_i|$ .

•

$$\begin{aligned} |D| &\leq |D_a| + \sum |D_i| \leq |D^*| + O(\gamma(G) / \ln^{c-1} |G|) \\ &= \gamma(G)(1 + O(1 / \ln^{c-1} |G|)). \end{aligned}$$



## Connected Dominating Set

1. Similar idea.
2. In each cluster  $V_i$  find an optimal connected dominating set  $D_i$  which contains all **centers**.
3. Connect different sets in incident small clusters by paths of length at most 3.
4. Return the union.

### Why does it work?

- **Problem:** A connected dominating set in  $G$  does not need to be connected in  $G[V_i]$ .
- If it is disconnected then components of  $G[V_i]$  contain vertices from the border clusters of  $V_i$ .

- The number of components in each  $G[V_i]$  can be large.
- When we add all centers of border clusters the number of components is small.
- We can connect the components using a small number of short paths.

**Lemma 5** *For any dominating set  $D$  there is a dominating set  $D'$  which contains  $D$ , contains centers, and each  $G[D' \cap V_i]$  is connected.*

## Conclusions

Distributed almost exact approximations for domination problems for minor-closed families.