2-Factors in Dense Bipartite Graphs

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Abstract

An *n-ladder* is a balanced bipartite graph with vertex sets $A = \{a, \dots, a_n\}$ and $B = \{b_1, \dots, b_n\}$ such that $a_i \sim b_j$ iff $|i - j| \leq 1$. We use techniques developed recently by Komlós, Sarkozy, and Szemerédi to show that if $G = (U, V, E)$ is a bipartite graph with $|U| = n = |V|$, with n sufficiently large, and the minimum degree of G is at least $\frac{n}{2} + 1$, then G contains an n-ladder. This answers a question of Wang.

Key words: bipartite graphs, blow-up lemma, cycles

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1 Introduction

A bipartite graph is said to *balanced* if it has the same number of vertices in each part. Let $G = (U, V, E)$ be a balanced bipartite graph with $|U| = n = |V|$. For the purposes of this article we shall call G *universal* if every balanced bipartite graph with n vertices in each part and maximum degree at most 2 can be embedded into G. Wang [8] conjectured that G is universal if the minimum degree $\delta(G)$ of G is at least $\frac{n}{2} + 1$. As Wang observed, the conjecture is best possible: The graph H obtained by adding a perfect matching between the larger parts of two copies of the complete bipartite graph $K_{m+1,m}$ satisfies $\delta(H) = m + 1 = \left\lceil \frac{n}{2} \right\rceil$, but does not contain the union $C_{2m} + C_{2m+2}$ of a $(2m)$-cycle with an $(2m + 2)$ -cycle.

Aigner and Brandt [1] proved the non-bipartite version of this conjecture: If H is a graph on n vertices with minimum degree $\delta(H) \geq \frac{2n-1}{3}$, then H con-

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tains every graph on n vertices with maximum degree at most 2. Again this is best possible, since m triangles cannot be embedded in the complete tripartite graph $K_{m-1,m+1,m+1}$. Fan and Kierstead [3] proved something stronger. In an attempt at proving Pósa's conjecture that every graph on n vertices with minimum degree at least $\frac{2}{3}n$ contains the square of a hamiltonian cycle, they showed that every graph H with $\delta(H) \geq \frac{2n-1}{3}$ contains the square Q of a hamiltonian path. Every graph on n vertices with maximum degree at most 2 can be embedded into Q. Then Komlós, Sarkozy, and Szemerédi [6] proved Pósa's conjecture using a wonderful new technique involving Szemerédi's Regularity Lemma [7] and their own Blow-Up Lemma [5].

Define an *n-ladder* to be the balanced bipartite graph L_n with vertex sets $A = \{a, \dots, a_n\}$ and $B = \{b_1, \dots, b_n\}$ such that $a_i \sim b_j$ iff $|i-j| \leq 1$. So L_n consists of two vertex disjoint n-paths $a_1b_2a_3b_4\dots$ and $b_1a_2b_3a_4,\dots$ together with *rungs* formed by the matching $\{a_1b_1,\dots,a_nb_n\}$. (Perhaps, twisted ladder would be a more descriptive term.) The set of *ends* of L_n is the set $\{a_1, b_1, a_n, b_n\}$. It is easy to see that the graph L_n is universal. For example, an even cycle $x_1y_1,\dots x_ky_k$ can be embedded into the beginning of L_n by

$$x_1y_1x_2\dots x_ky_k \mapsto a_1b_2a_3\dots s_kt_k, s_{k-1}, t_{k-1}, \dots a_2, b_1,$$

where $(s_k,t_k) = (a_k,b_k)$, if k is odd and $(s_k,t_k) = (b_k,a_k)$ otherwise. We shall apply the methods of Komlós, Sarkozy, and Szemerédi to prove the following strengthening of Wang's conjecture for sufficiently large n.

Theorem 1 *For sufficiently large n, every balanced bipartite graph $G = (U,V,E)$ with $|U| = n = |V|$ and $\delta(G) \geq \frac{n}{2}+1$ contains a spanning ladder.*

The rest of this article is devoted to a proof of Theorem 1. This involves two parts. First, in Section 2 we identify a particular extremal configuration and use standard graph theoretical methods to show that if G contains this configuration, then G contains a spanning ladder. Then in Section 4, we use the Regularity Lemma and the Blow-Up Lemma to show that if G does not contain the extremal configuration, then G contains a spanning ladder. In Section 3 we review the Regularity and Blow-Up Lemmas. In the remainder of this section we review our notation and work out some easy details of the theory of dense bipartite graphs.

For a positive integer n, let $[n]$ denote the set $\{1,2,\dots,n\}$. Let $\deg(v)$ denote the degree of the vertex v and let $\deg(v,X)$ denote the number of neighbors of v in the set X. For two disjoint, nonempty sets of vertices U and W, let $e(U,W)$ denote the number of edges with end points in both U and W.

We will need Lemma 3 for our argument and the rest of this section serves as a warm-up. The next lemma is due to Bondy and Chvátal [2]

Lemma 2 *Let* $P = x_1y_1 \dots x_sy_s$ *be an even path of a bipartite graph* $H = (X, Y, E)$ *with* $s \geq 2$. *If* $\deg(x_1, P) + \deg(y_s, P) \geq s + 1$ *then* G *has a cycle* C *with* $V(C) = V(P)$.

PROOF. By the pigeon hole principle, there exists $i \in [s]$ such that $x_1 \sim y_i$ and $y_s \sim x_i$. Then $x_1y_1 \dots x_iy_sx_sy_{s-1}x_{s-1} \dots y_i$ is the desired cycle.

Lemma 3 *Let* $H = (X, Y, E)$ *be a connected bipartite graph with* $|X| = |Y| = n$. *If* $\delta(H) \geq k$ *then* H *contains a path on* $\min\{4k-1, 2n\}$ *vertices.*

PROOF. Let P be the longest path in H and assume that the length of P is less than $\min\{4k-1, 2n\}$.

Case 1: $P = x_1y_1 \dots x_sy_s$ where $x_i \in X, y_i \in Y$. Then $d(x_1, P) + d(y_s, P) \geq 2k > s$. Thus, by Lemma 2, there is a cycle C with $V(C) = V(P)$. Since $s < n$ there is a vertex v not on C. Since H is connected one can obtain a longer path starting from v and using all of C.

Case 2: $P = x_1y_1 \dots x_sy_sx_{s+1}$. First we claim that H contains a cycle on $2s$ vertices. Indeed, since P is maximal, x_1 and x_{s+1} have all their neighbors on P. Since $\deg(x_1) + \deg(x_{s+1}) \geq 2k > s + 1$, there exists $i \in [s-1]$ such that $x_1 \sim y_{i+1}$ and $x_{s+1} \sim y_i$. This gives a cycle $x_1y_1 \dots x_iy_ix_{s+1}y_sx_s \dots y_{i+1}$. Assume now that $C = x_1y_1 \dots x_sy_s$ is a cycle. Suppose that v is a vertex not on C. Then v is only adjacent to vertices on C, since otherwise there is a path longer than P. Since $s < n$, there exist $x \in X \setminus V(C)$ and $y \in Y \setminus V(C)$. Since $\deg(x) + \deg(y) \geq 2k > s$, there exists $i \in [s]$ such that $x \sim y_i$ and $y \sim x_i$. This yields a path on $2(s+1)$ vertices and a contradiction.

Notice that it now follows that a balanced bipartite graph with n vertices in each part and minimum degree greater than $\frac{n}{2}$ is hamiltonian: By the degree condition, it is connected. By Lemma 3 it has a hamiltonian path. So by the degree condition and Lemma 2, it has a hamiltonian cycle.

2 The Extremal Case

We say that $G = (U, V, E)$ is *α-splittable* if there exist $X \subset U$ and $Y \subset V$ such that

$$|X| = \left(\frac{1}{2} - \alpha\right) n = |Y| \quad \text{and} \quad e(X, Y) \leq \alpha n^2.$$

In this case we say that X and Y are an α-splitting of G.

Lemma 4 *For $\alpha > 0$ sufficiently small, in particular $\alpha \leq \frac{1}{640000}$, if $G = (U, V, E)$ is a balanced bipartite graph with n vertices in each part that is α -splittable and $\delta(G) \geq \frac{n}{2} + 1$, then G contains a spanning ladder.*

PROOF. Let X and Y be an α-splitting of G. Our first goal is to partition the vertices of G into two almost complete, bipartite, spanning subgraphs with about $\frac{n}{2}$ vertices in each part. Let $\overline{X} = U \setminus X$ and $\overline{Y} = V \setminus Y$. Let $S = \left\{x \in X : \deg\left(x, \overline{Y}\right) < \left(\frac{1}{2} - \sqrt{\alpha}\right) n\right\}$. For each $x \in S$, $\deg(x, Y) \geq \sqrt{\alpha} n$. Thus $|S| \leq \sqrt{\alpha} n$, since

$$|S| \sqrt{\alpha} n \leq \sum_{x \in S} \deg(x, Y) \leq e(X, Y) \leq \alpha n^2.$$

Counting the edges of the complement $\overline{G}$ of G from X to $\overline{Y}$ we get:

$$\begin{aligned} e_{\overline{G}}\left(X, \overline{Y}\right) &= |X| \left|\overline{Y}\right| - \left(\sum_{x \in X} \deg(x) - e(X, Y)\right) \\ &\leq \left(\frac{1}{2} - \alpha\right)\left(\frac{1}{2} + \alpha\right) n^2 - \left(\frac{1}{2} - \alpha\right) \frac{n^2}{2} + \alpha n^2 \\ &\leq \frac{3}{2} \alpha n^2 \end{aligned}$$

Let $T = \left\{y \in \overline{Y} : \deg(y, X) < \left(\frac{1}{2} - \alpha - \sqrt{\alpha}\right) n\right\}$. For each $y \in T$, $\deg_{\overline{G}}(y, X) \geq \sqrt{\alpha} n$. Thus $|T| \leq \frac{3}{2} \sqrt{\alpha} n$, since

$$|T| \sqrt{\alpha} n \leq \sum_{y \in T} \deg_{\overline{G}}(y, X) \leq e_{\overline{G}}\left(X, \overline{Y}\right) \leq \frac{3}{2} \alpha n^2.$$

Thus we can choose $X_1 \subset X \setminus S$ and $Y_1 \subset \overline{Y} \setminus T$ such that

(1) $|X_1|, |Y_1| = \left(\frac{1}{2} - 2\sqrt{\alpha}\right) n$ and
(2) $\deg(x_1, Y_1), \deg(y_1, X_1) \geq \left(\frac{1}{2} - 4\sqrt{\alpha}\right) n$, for all $x_1 \in X_1$, $y_1 \in Y_1$.

Similarly, we can choose $X_2 \subset \overline{X}$ and $Y_2 \subset Y$ such that

(1) $|X_2|, |Y_2| = \left(\frac{1}{2} - 2\sqrt{\alpha}\right) n$ and
(2) $\deg(x, Y_2), \deg(y, X_2) \geq \left(\frac{1}{2} - 4\sqrt{\alpha}\right) n$, for all $x \in X_2$ and $y \in Y_2$.

Let $X_0 = U \setminus (X_1 \cup X_2)$ and $Y_0 = V \setminus (Y_1 \cup Y_2)$. Then $|X_0| = 4\sqrt{\alpha} n = |Y_0|$. Let $\beta = 4\sqrt{\alpha}$. Choose partitions $\{S_1, S_2\}$ of X_0 and $\{T_1, T_2\}$ of Y_0 such that

$g = ||S_1| - |T_1||$ is as small as possible subject to the condition that

$$\deg(s_i, Y_i), \deg(t_i, X_i) \geq 8\beta n$$

for all $s_i \in S_i$, $t_i \in T_i$, and $i \in [2]$. For $i \in [2]$, let $U_i = X_i \cup S_i$, $V_i = Y_i \cup T_i$, $G_i = G[U_i, V_i]$, the subgraph of G induced by $U_i \cup V_i$, and $n_i = |U_i|$. This accomplishes our first goal: All but at most βn vertices of G_i have degree at least $\left(\frac{1}{2} - \beta\right) n$ in G_i; moreover each exceptional vertex has at least $8\beta n$ nonexceptional neighbors. Without loss of generality, $\max\{|S_1|, |S_2|, |T_1|, |T_2|\} = |S_2|$. So $g = |T_1| - |S_1| = |S_2| - |T_2|$. Since

$$\begin{aligned} |U_2| \geq |V_1| &= n - |V_2| \\ |U_2| - \frac{n}{2} &\geq \frac{n}{2} - |V_2| \end{aligned}$$

it follows that

$$|U_2| - \frac{n}{2} \geq \frac{g}{2}.$$

So every vertex in V_1 has at least $\frac{g}{2} + 1$ neighbors in U_2 . Also if $x \in X_i$ and $u \in U_i$, then x and u have at least $7\beta n$ common neighbors in Y_i. Similarly, if $y \in Y_i$ and $v \in V_i$, then y and v have at least $7\beta n$ common neighbors in X_i.

Our next goal is to construct two disjoint ladders L and L^* (L^* might be empty) and possibly move some vertices from G_1 to G_2 such that:

(1) The first rung of L is in G_1 and the last rung of L is in G_2.
(2) The first rung of L^* is in G_2.
(3) $G_i \setminus (L \cup L^*)$ is balanced for $i \in \{1, 2\}$.
(4) $|L| \leq 6$ and $|L^*| \leq 6g + 4$

A 2-ladder is called a *crossing* L_2 if it has one vertex in each of U_1, U_2, V_1, and V_2.

Case 1: $g = 0$ and G contains a crossing L_2. Let $L = L_2$ and $L^* = \emptyset$.

Case 2: $g = 0$, but G does not contain a crossing L_2. Then for every $u_1 \in U_1$ and $u_2 \in U_2$, there exists $i \in [2]$ such that all common neighbors of u_1 and u_2 are in V_i. Since $\delta(G) \geq \frac{n}{2} + 1$, u_1 and u_2 have at least two common neighbors in V_i. Similarly, for every $v_1 \in V_1$ and $v_2 \in V_2$, there exists $i \in [2]$ such that all (at least two) common neighbors of v_1 and v_2 are in V_i.

Fix $x, x' \in X_1$ and $y \in Y_1$ with $x \sim y \sim x'$. Since $n_1 \leq n_2$, there exists $u \in U_2$ such that $y \sim u$. Since $y \in V_1$ is a common neighbor of $x \in U_1$ and $u \in U_2$, there exists another common neighbor $v \in V_1$ of x and u. Let $y' \in Y_2$ be a neighbor of u. Since $u \in U_2$ is a common neighbor of $v \in V_1$ and $y' \in V_2$, there exists another common neighbor u' of v and y' in U_2. So $L = (x, y, u, v, u', y')$ is a 3-ladder.

Now removing L leaves G_1 and G_2 unbalanced, since L has an extra vertex from each of U_2 and V_1. We correct this problem by constructing another crossing ladder L^*. Choose, in order, distinct $v' \in V_2$, $x'' \in X_2 \setminus \{u, u'\}$, and $v'' \in V_2$ such that $x' \sim v' \sim x'' \sim v'' \sim x'$. Note that $y' \notin \{v', v''\}$, since otherwise $\{x', y, u, y'\}$ is a crossing L_2. So $L^* = (x'', v'', x', v')$ is a 2-ladder disjoint from L and (1) - (3) are satisfied.

Case 3: $g > 0$. Since we could not move any vertices from U_2 to U_1 to decrease the gap g, $\deg(u, Y_1) < 8\beta n$, and so $\deg(u, Y_2) \geq \left(\frac{1}{2} - 9\beta\right) n$, for every vertex $u \in U_2$. It follows that any two vertices in U_2 have lots of common neighbors in Y_2. Since $\frac{g}{2} > 0$, every vertex in Y_1 has at least two neighbors in U_2 . Since $|U_2| < 2\,|Y_1|$, some vertex $a_2 \in U_2$ has two neighbors $b_1, b_2 \in Y_1$. Let $a_3 \in U_2$ be another neighbor of b_2. Finally let $a_1 \in X_1$ be a common neighbor of b_1 and b_2 and $b_3 \in Y_2$ be a common neighbor of a_2 and a_3. Then $L = (a_1, b_1, a_2, b_2, a_3, b_3)$ is a 3-ladder with first rung $a_1 b_1$ and last rung $a_3 b_3$.

Case 3a: $g = 1$. Letting L^* be empty, we are done.

Case 3b: $g = 2$ Since $|Y_1| > 16\beta n \geq e\left(\{a_2, a_3\}, Y_1\right)$ there exists $b_5 \in Y_1$ with two neighbors $a_4, a_5 \in U_2 \setminus \{a_2, a_3\}$. Let $b_4 \in V_2$ be a common neighbor of a_4 and a_5. Then $L^* = (a_4, b_4, a_5, b_5)$ is a 2 -ladder disjoint from L and (1) - (3) are satisfied.

Case 3c: $g > 2$. We still have the link L from G_1 to G_2. To obtain two balanced bipartite graphs, we will construct a ladder L^* that contains $g-1$ more vertices from V_1 than V_2. A *triple matching* is a set of vertex disjoint 3-stars $Q_i = (v_i'', u_i', u_i'', u_i''')$ with root $v_i'' \in V_1 \setminus \{b_1, b_2\}$ and leaves $u_i', u_i'', u_i''' \in U_2 \setminus \{a_2, a_3\}$. We claim that there exists a triple matching of size $g - 1$. Otherwise let $M = \{Q_1, \ldots Q_k\}$ be a maximum triple matching of size k with roots $R = \{v_i'' : i \in [k]\}$ and leaves $Z = \{u_i', u_i'', u_i''' : i \in [k]\}$. Since M is maximum, each vertex $v \in V_1 \setminus (R \cup \{b_1, b_2\})$ has at most two neighbors in $U_2 \setminus (Z \cup \{a_2, a_3\})$. Thus $\deg\left(v, Z \cup \{a_2, a_3\}\right) \geq \frac{g}{2} - 1$ and by the pigeon hole principle there exists a vertex $u \in Z \cup \{a_2, a_3\}$ such that

$$\deg(u, V_1) \geq \frac{\left(\frac{g}{2} - 1\right)\left(|V_1| - k - 1\right)}{3k + 2} \geq \frac{\frac{g-2}{2}\left(\frac{1}{2} - \beta\right) n}{3(g-2) + 2} \geq \frac{\left(\frac{1}{2} - \beta\right) n}{10} > 9\beta n$$

which is a contradiction. Next we construct the ladder L^* that contains each Q_i as follows. Choose distinct $v_1', v_1''', \ldots, v_{g-1}', v_{g-1}''' \in V_2 \setminus \{b_3\}$ such that v_1' is adjacent to u_1' and u_1'', v_i' is adjacent to u_{i-1}'', u_{i-1}''' , and u_i', v_i''' is adjacent to u_i'', u_i''', and u_{i+1}', and v_{g-1}''' is adjacent to u_{g-1}'' and u_{g-1}'''. This is possible since

$3g \leq 3\beta n \leq \left(\frac{1}{2} - 27\beta\right) n$. Then

$$L^* = \left(u_1', v_1', u_1'', v_1'', u_1''', v_1''', \ldots, u_{g-1}', v_{g-1}', u_{g-1}'', v_{g-1}'', u_{g-1}''', v_{g-1}'''\right)$$

and the conditions (1) - (4) are satisfied.

Finally we finish our construction of a spanning ladder by constructing, for $i \in \{1, 2\}$, ladders L^i in $G_i \backslash (L \cup L^*)$ such that the last rung of L^1 and the first rung of L form a 2-ladder, the last rung of L and the first rung of L^2 form a 2-ladder, and the last rung of L^2 and the first rung of L^* form a 2-ladder. Since the construction of L^1 is similar, but easier, we will only give the construction of L^2.

Note that $G' = G_2 \setminus (L \cup L^*)$ has between $\left(\frac{1}{2} - 3\beta\right) n$ and $\left(\frac{1}{2} + \beta\right) n$ vertices in each part. Of the vertices in one part, at most βn are not adjacent to at least $\left(\frac{1}{2} - 4\beta\right) n$ vertices, and even these vertices have degree at least $5\beta n$ in G'. Call these the exceptional vertices. Write G' as $G' = (U', V', E')$, let $S' \subset U'$ and $T' \subset V'$ be the small subsets of exceptional vertices, and let $X' = U' \setminus S'$ and $V' \setminus T'$ be the large subsets of normal vertices. With less effort than above we can find a triple matching whose roots are the vertices of $S' \cup T'$ and whose leaves come from $X' \cup Y'$. Each 3-star from this matching can be extended to a 3-ladder by adding vertices from $X' \cup Y'$. Label these 3-ladders by $N^1, \ldots, N^s$. Since

$$\frac{1}{2}\left(\frac{1}{2} + \beta\right) n < \left(\frac{1}{2} - 4\beta\right) n - 6\beta n,$$

the remaining vertices from $X' \cup Y'$ can be matched, say by $M = \{x_1 y_1, \ldots, x_t y_t\}$. Moreover we can specify the first and last edge of this matching so that the last rung of L forms a 2-ladder with $x_1 y_1$ and the first rung of L^* forms a 2-ladder with $x_t y_t$. Define an auxiliary graph A on the vertex set $\{N^1, \ldots, N^s\} \cup M$. We treat the edges of M as 1-ladders. Two vertices of A are adjacent if the first and last rungs of one form 2-ladders with the first and last rungs of the second. The degree of this graph is at least $\left(\frac{1}{2} - 16\beta\right) n \geq \frac{s+t}{2} + 1$, so it has a hamiltonian path from $x_1 y_1$ to $x_t y_t$. Clearly this path corresponds to the desired ladder.

3 The Regularity and Blow-Up Lemmas

In this section we review the Regularity and Blow-Up Lemmas. Let $H = (V, E)$ be a simple graph on n vertices. For two disjoint, nonempty subsets U and W

of V, define the density of the pair (U,W) as

$$d(U,W)=\frac{e(U,W)}{|U||W|}.$$

Definition 5 *A pair (U,W) is called* ε-regular *if for every $U'\subset U$ with $|U'|\geq\varepsilon|U|$ and every $W'\subset W$ with $|W'|\geq\varepsilon|W|$, $|d(U',W')-d(U,W)|\leq\varepsilon$. The pair (U,W) is* (ε,δ)-super-regular *if it is ε-regular and for all $u\in U$, $\deg(u,W)\geq\delta\,|W|$ and for all $w\in W$, $\deg(w,U)\geq\delta\,|U|$.*

First we note the following three facts that we will need.

Lemma 6 *If (U,W) is an ε-regular pair with density d, then for any $Y\subset W$ with $|Y|\geq\varepsilon\,|W|$ there are at most $\varepsilon|U|$ vertices $u\in U$ such that $deg(u,Y)<(d-\epsilon)|Y|$.*

Lemma 7 (Slicing Lemma) *Let (U,W) be an ε-regular pair with density d, and for some $v>\varepsilon$ let $U'\subset U$, $W'\subset W$, with $|U'|\geq v|U|$, $|W'|\geq v|W|$. Then (U',W') is an ε'-regular pair of density d' where $\varepsilon'=\max\{\frac{\varepsilon}{v},2\varepsilon\}$ and $d'>d-\varepsilon$.*

Lemma 8 *Let (U,W) be an ε-regular pair. Suppose that $U'=U\cup S$ and $W'=W\cup T$, where $|S|\leq\mu\,|U|$, $|T|\leq\mu\,|W|$, $S\cap W'=\emptyset=T\cap U'$, and $0<\mu<\varepsilon$. Then (U',W') is an ε'-regular pair, where $\varepsilon'=\max\left\{\frac{\mu}{\varepsilon},6\varepsilon\right\}$.*

Definition 9 *Partition $V_0\cup V_1\cup\ldots\cup V_t$ of the vertex set of $G=(V,E)$ is called ε-regular if the following conditions are satisfied.*

(1) $|V_0|\leq\epsilon|V|$.
(2) For all $1\leq i,j\leq t$, $|V_i|=|V_j|$.
(3) All but at most εt^2 of pairs (V_i,V_j), $1\leq i,j\leq t$, are ε-regular.

The parts of the partition are called *clusters*. Note that the cluster V_0 plays a distinguished role in the above definitions and is usually called the exceptional cluster (or class). Our main tool in the proof will be the regularity lemma of Szemerédi [7] which asserts that for every $\varepsilon>0$ every graph which is large enough admits an ε -regular partition into a bounded number of clusters.

Lemma 10 (Regularity Lemma) *For every $\varepsilon>0$ and every positive integer l there exist $N=N(\varepsilon,l)$ and $L=L(\varepsilon,l)$ such that every graph with at least N vertices admits an ε-regular partition $V_0\cup V_1\cup\ldots V_t$ with $l\leq t\leq L$.*

In addition, we shall use the Blow-up Lemma of Komlós, Sarkozy, and Szemerédi [5].

Lemma 11 (Blow-Up Lemma) *Given $\delta>0$ and $\Delta>0$ there exists an $\varepsilon>0$ such that the following holds. Let $P=(W_1,W_2)$ be an (ε,δ)-super-*

regular pair with $|W_1| = n_1$ *and* $|W_2| = n_2$. *If a graph* $H = (A_1, A_2)$ *with maximum degree* $\Delta(H) < \Delta$ *is embeddable into the complete bipartite graph* K_{n_1,n_2} *then it is also embeddable into* P. *Moreover, given in addition* $\beta > 0$, *there exist* γ *and* ε *such that the following stronger statement is true. For all* γn_i*-subsets* $A_i' \subset A_i$ *and functions* $f_i : A_i' \to \binom{W_i}{\beta n_i}$, $i = 1, 2$, H *can be embedded into* P *so that the image of each* $a_i \in A_i'$ *is in the set* $f_i(a_i)$.

4 The Non-Extremal Case

In this section we complete the proof of our main theorem by proving the following lemma.

Lemma 12 *For any* $\alpha > 0$ *and sufficiently large* n, *if* $G = (U, V, E)$ *is a balanced bipartite graph with* n *vertices in each part that is not* α*-splittable and* $\delta(G) \geq \frac{n}{2}$, *then* G *contains a spanning ladder.*

PROOF. Fix α. Let $\delta_2 \leq \frac{\alpha}{4}$, $\delta_1 \leq \frac{\delta_2^2}{64}$, $\delta \leq \frac{\delta_1}{2}$, $\beta = \frac{\delta}{2}$, and $\Delta = 4$. Now, for this choice of δ, β, and Δ, choose γ and $\varepsilon < \delta^3$ so that the strong conclusion of the Blow-Up Lemma holds. Let $\varepsilon_1 \leq \left(\frac{\varepsilon}{6}\right)^4$ and $l \geq \frac{8}{\varepsilon_1}$. By the Regularity Lemma (applied to the graph G with $2n$ vertices) there exists N and L such that if $2n > N$, then there exists an $\frac{\varepsilon_1}{4}$-regular partition of $G = (U, V, E)$ with between l and L clusters. We will also require that $n > \frac{16L}{\gamma}$. Note that since in the proof of the Regularity Lemma a given partition is refined to yield a regular partition, we can start initially with the partition $U \cup V$ and end up with a partition of the form $\{V_0, V_1, \ldots V_{k_1}, U_0, U_1, \ldots, U_{k_2}\}$, where $\{V_0, V_1, \ldots V_{k_1}\}$ is a partition of V, $\{U_0, U_1, \ldots, U_{k_2}\}$ is a partition of U, and the exceptional class has the form $V_0 \cup U_0$. If $k_1 > k_2$ then we add $k_1 - k_2$ clusters to the exceptional cluster V_0 so that we can assume that $k = k_1 = k_2$. Then $|V_0| \leq \frac{\varepsilon_1}{2} n$. Define the cluster graph C_G as follows: $V(C_G) = \{V_1, \ldots V_k, U_1, \ldots, U_k\}$ and V_i is adjacent to U_j iff (V_i, U_j) is an $\frac{\varepsilon_1}{4}$-regular pair and $d(V_i, U_j) \geq \delta_1$. For a cluster W, let $\mathrm{irrdeg}(W)$ denote the number of clusters W' such that (W, W') is $\frac{\varepsilon_1}{4}$-irregular. By the $\frac{\varepsilon_1}{4}$-regularity of the partition, there are at most $\frac{\varepsilon_1}{4}(2k)^2 = \varepsilon_1 k^2$ pairs of clusters that are $\frac{\varepsilon_1}{4}$-irregular. Thus there are at most $\sqrt{\varepsilon_1} k$ clusters W for which $\mathrm{irrdeg}(W)$ is at least $\sqrt{\varepsilon_1} k$. Since we can add these clusters to $V_0 \cup U_0$, we may assume that for every cluster W,

$$\begin{aligned} &\mathrm{irrdeg}(W) < \sqrt{\varepsilon_1} k, \\ &|V_0|, |U_0| \leq 2\sqrt{\varepsilon_1} n, \quad \text{and} \\ &(1 - \varepsilon_1)\frac{n}{k} \leq |W| \leq \frac{n}{k} \end{aligned}$$

.

Claim 1. Set $\rho = \delta_1 + 4\sqrt{\varepsilon_1}$. Then $\delta\left(C_G\right) \geq (\frac{1}{2} - \rho)k$.

PROOF. Assume that there is a cluster $W \in V(C_G)$ such that $\deg(V_i) < (1/2 - \rho)k$. Then W has at most $(1/2 - \rho)k\,|W|^2$ edges to vertices in adjacent clusters, $\sqrt{\varepsilon_1}k\,|W|^2$ edges to vertices in clusters forming irregular pairs with W, $\delta_1 k\,|W|^2$ edges to clusters forming low density pairs with W, and $2\sqrt{\varepsilon_1}n\,|W|$ edges to the exceptional cluster. So

$$e(W, V(G)) < (\frac{1}{2} - \rho + 3\sqrt{\varepsilon_1} + \delta_1)\frac{n^2}{k} \leq \left(\frac{1}{2} - \sqrt{\varepsilon_1}\right)\frac{n^2}{k}. \tag{1}$$

On the other hand, using the fact that the minimum degree of G is at least $\frac{n}{2}$, we have

$$e(W, V(G)) \geq |W|\,\frac{n}{2} \geq \frac{(1-\varepsilon_1)\,n^2}{2k} \geq \left(\frac{1}{2} - \frac{1}{2}\varepsilon_1\right)\frac{n^2}{k}. \tag{2}$$

Clearly (1) and (2) are not possible at the same time. Therefore, the minimum degree of C_G is at least $(1/2 - \rho)k$.

Claim 2. The cluster graph has a path of length at least $4\left(\frac{1}{2} - \rho\right)k - 1$.

PROOF. Let $\rho' = \rho + \varepsilon_1$. We first show that C_G is connected. Suppose to the contrary, that C_G is disconnected. Since $\delta\left(C_G\right) \geq (\frac{1}{2} - \rho)k$, each component of C_G has at least $(\frac{1}{2} - \rho)k$ clusters in each part. Thus the union of the clusters in one part of a component has at least

$$(\frac{1}{2} - \rho)k\,(1 - \varepsilon_1)\,\frac{n}{k} \geq \left(\frac{1}{2} - \rho - \frac{1}{2}\varepsilon_1\right)n$$

vertices. Let X be a set of $\left(\frac{1}{2} - \rho'\right)n$ vertices from one part of one component of C_G and let Y be a set of $\left(\frac{1}{2} - \rho'\right)n$ vertices from the other part of another component of C_G. There are only two kinds of edges between X and Y: (1) Edges between clusters that form irregular pairs and (2) edges between clusters that form low density pairs. There are at most $\varepsilon_1 k^2\left(\frac{n}{k}\right)^2 = \varepsilon_1 n^2$ of the first kind and at most $\delta_1 n^2$ of the second kind. Thus $e\left(X, Y\right) \leq \rho' n^2$ and G is ρ'-splittable. Since $\rho' < \alpha$ and G is not α-splittable, this is a contradiction. We conclude that C_G is connected. By Lemma 3 there exists a path on $2k - 4\rho k - 1$ clusters in C_G.

Let $P = U_1 V_1 \dots U_r V_r$ be a path in C_G of length $2r = 2k - 4\rho k - 2$. Then consecutive clusters in P form ε_1-regular pairs with density at least δ_1. Reassign one vertex from U_1 to U_r. Add the vertices in the clusters that are not on P to the exceptional classes V_0 and U_0. Now

$$|V_0|, |U_0| \leq 3\delta_1 n.$$

This is a major change in the size of $|V_0 \cup U_0|$.

Our next goal is to reassign the exceptional vertices to regular clusters in P. This is a process that we will need to use again with different parameters, so we introduce general parameters σ_1 and σ_2. (In our first application of the process we will set $\sigma_1 = 3\delta_1$ and $\sigma_2 = \delta_2$.) We would like to do this so that $|U_i| - |V_i|$ remains constant, there are only a small number, at most $\sqrt{\sigma_1}\frac{n}{k}$, of vertices reassigned to any one cluster, and so that any vertex reassigned to a cluster U_i (V_i) has a relatively large degree, at least $\sigma_2 \frac{n}{k}$, to V_i (U_i). During this process we may also reassign some vertices from one cluster in P to another cluster in P. More formally, for each $i \in [r]$, let U_i^0 (V_i^0) denote the elements of U_i (V_i) at the start of this process, U_i' (V_i') denote the set of vertices reassigned to U_i (V_i) and U_i^* (V_i^*) denote the set of vertices of U_i (V_i) that have not been reassigned. So $U_i^0 \setminus U_i^*$ is the set of vertices reassigned from U_i. We will preserve the following conditions:

(1) If v is reassigned to V_i then $\deg(v, U_i^0) \geq \sigma_2 \frac{n}{k}$ and if u is reassigned to U_i then $\deg(u, V_i^0) \geq \sigma_2 \frac{n}{k}$.
(2) For every $2 \leq i \leq r-1$, $|U_i^* \cup U_i'| = |V_i^* \cup V_i'|$, $|U_1^* \cup U_1'| + 1 = |V_1^* \cup V_1'|$, and $|U_r^* \cup U_r'| = |V_r^* \cup V_r'| + 1$.
(3) For each $i \in [r]$, $|V_i'|, |U_i'| \leq \sqrt{\sigma_1}\frac{n}{k}$.
(4) For each $i \in [r]$, $|V_i^0 \setminus V_i^*|, |U_i^0 \setminus U_i^*| \leq \sqrt{\sigma_1}\frac{n}{k}$.

We group the exceptional vertices into pairs $(v, u) \in V_0 \times U_0$ and order the pairs arbitrarily. We then reassign these vertices one pair at a time as follows. Call a cluster W *full* if there are $\sqrt{\sigma_1}\frac{n}{k}$ vertices reassigned to W. Consider the next pair (v, u). Try to choose $i, j \in [r]$ such that

a. neither V_i, U_i, nor U_j is full,
b. $\deg\left(v, U_i^0\right) \geq \sigma_2 \frac{n}{k}$ and $\deg\left(u, V_j^0\right) \geq \sigma_2 \frac{n}{k}$, and
c. if $i \neq j$ then $e\left(U_j^0, V_i^0\right) \geq 2\sigma_2 \left(\frac{n}{k}\right)^2$.

Then reassign u to U_j, v to V_i, and if $i \neq j$, then pick $u' \in U_j^*$ with $\deg\left(u', V_i^0\right) \geq \sigma_2 \frac{n}{k}$, and reassign u' to U_i. If we can perform this operation for each pair, then we will succeed in reassigning the exceptional vertices while maintaining (1)-(4). Notice in particular that since a vertex is only reassigned from a regular cluster if another vertex is reassigned to that cluster, (4) follows from (3).

Claim 3. If

$$|U_0|, |V_0| \le \sigma_1 n \quad \text{and} \quad \varepsilon_1 < \sigma_1 < 4\sqrt{\sigma_1} \le \sigma_2 < 4\sigma_2 \le \alpha,$$

then for each pair (v, u) we can choose $i, j \in [r]$ satisfying (1) - (3) above.

PROOF. Consider the next pair (v, u). Let $N'(v) = \{i : deg(v, U_i^0) \ge 2\sigma_2 \frac{n}{k}\}$ and $N'(u) = \{i : deg(u, V_i^0) \ge 2\sigma_2 \frac{n}{k}\}$. Since v is adjacent to at most $\frac{n}{k}$ vertices in each U_i^0 with $i \in N'(v)$, at most $\sigma_1 n$ vertices in U_0, and at most $2\sigma_2 n$ other vertices,

$$|N'(v)| \frac{n}{k} + \sigma_1 n + 2\sigma_2 n \ge \deg(v) \ge \frac{n}{2}.$$

This yields

$$|N'(v)| \ge (1 - \sigma_1 - 2\sigma_2) k \ge (1/2 - 3\sigma_2) k.$$

Similarly $|N'(u)| \ge (1/2 - 3\sigma_2) k$.

Let

$$Y = \bigcup_{i \in N'(u)} V_i^0 \quad \text{and} \quad X = \bigcup_{i \in N'(v)} U_i^0.$$

Recalling that $\alpha \ge 4\sigma_2$,

$$|Y| \ge |N'(v)| (1 - \varepsilon_1) \frac{n}{k} \ge (1/2 - 3\sigma_2) k (1 - \varepsilon_1) \frac{n}{k} \ge \left(\frac{1}{2} - \alpha\right) n.$$

Similarly, $|X| \ge \left(\frac{1}{2} - \alpha\right) n$. Since G is not α-splittable, $e(Y, X) \ge \alpha n^2 \ge 4\sigma_2 n^2$.

At most $\sigma_1 n$ pairs of exceptional elements have been reassigned. Each time a pair is reassigned there are at most two indices i such that $|V_i'|$ or $|U_i'|$ increases and for all indices j neither $|V_j'|$ nor $|U_j'|$ ever increases by more than one. Thus we have created at most

$$2\sqrt{\sigma_1} k = \frac{2\sigma_1 n}{\sqrt{\sigma_1} \frac{n}{k}}$$

pairs (V_i^0, U_i^0) such that either V_i is full or U_i is full. The number of edges of G incident to vertices of these pairs is at most $4\sqrt{\sigma_1} n^2$ and there are less than

$2\sigma_2 n^2$ edges of G between the pairs (U_i^0, V_j^0) for which $e(U_i^0, V_j^0) < 2\sigma_2 \left(\frac{n}{k}\right)^2$. Since

$$(2\sigma_2 + 4\sqrt{\sigma_1})n^2 \leq 3\sigma_2 n^2 < \alpha n^2 \leq e(Y, X),$$

there must exist $i \in N'(v)$ and $j \in N'(u)$ such that neither U_i, V_i, nor U_j are full and $e\left(U_j^0, V_i^0\right) \geq 2\sigma_2 \left(\frac{n}{k}\right)^2$.

Notice that the hypothesis of Claim 3 is satisfied with $\sigma_1 = 3\delta_1$ and $\sigma_2 = \delta_2$. After reassigning the exceptional vertices, the path clusters are partitioned by $V_i = V_i' \cup V_i^*$ and $U_i = U_i' \cup U_i^*$. For each $i \in [r]$, U_i^* is a large subset of U_i^0 and V_i^* is a large subset of V_i^0 and so by the Slicing Lemma the pairs (U_i^*, V_i^*) and $\left(U_i^*, V_{i+1}^*\right)$ maintain most of their regularity. The same cannot be said for U_i' and V_i', but to compensate, U_i' and V_i' are small subsets ($\leq \sqrt{3\delta_1}\frac{n}{k}$) such that $\deg(u, V_i) \geq \delta_2 \frac{n}{k}$ and $\deg(v, U_i) \geq \delta_2 \frac{n}{k}$ are extra large for $u \in U_i'$ and $v \in V_i'$. Our next goal is to hide the exceptional vertices in a small ladder.

Claim 4. For each $i \in [r]$ there exists a ladder $L^i \subset V_i \cup U_i$ such that:
(1) $V_i' \cup U_i' \subset V(L^i)$.
(2) $|V(L^i)| \leq 32\sqrt{\delta_1}\frac{n}{k}$.
(3) Each of the ends of L^i has at least $\frac{\delta_1 n}{2k}$ neighbors in $(V_i \cup U_i) \setminus L^i$.

PROOF. Let $w_1, w_2, \ldots, w_t$ be an ordering of $V_i' \cup U_i'$. Then $t \leq 4\sqrt{\delta_1}\frac{n}{k} = \frac{\delta_2 n}{16k}$. Suppose that we have constructed a ladder $L \subset V_i \cup U_i$ on $8s$ vertices ($0 \leq s \leq t$) that contains exactly the first s vertices of $V_i' \cup U_i'$, satisfies (3), and has first rung $u'v'$ and last rung $u''v''$. Without loss of generality, assume that $w_{s+1} \in U_i'$. We first extend L to L' by attaching a 3-ladder $(a, b, a', b', w_{s+1}, v)$, with $a, a' \in U_i^* \setminus L$ and $b, b', v \in V_i^* \setminus L$, to the end of L. Using the regularity of the pair (U_i^0, V_i^0), Lemma 6 , and the fact that

$$\deg(w_{s+1}, V_i^* \setminus L) \geq \delta_2 \frac{n}{k} - 8t \geq \frac{\delta_2 n}{2k},$$

there exists $v \in V_i^* \cap N(w_{s+1})$ such that

$$\deg(v, U_i^* \setminus L) \geq (\delta_1 - \varepsilon_1)|U_i^* \setminus L| \geq \frac{\delta_1 n}{2k} + 5.$$

Let $A = N(v'') \cap (U_i^* \setminus L)$, $A' = N(v) \cap (U_i^* \setminus L,)$, $B = N(u'') \cap (V_i^* \setminus (L \cup \{v\}))$, and $B' = N(w_{s+1}) \cap U_i^* \setminus L$. Each of these sets has size at least $\frac{\delta_1 n}{2k}$. Using the regularity of the pair (U_i^0, V_i^0) and Lemma 6, almost all (all but at most $2\varepsilon_1 \frac{n}{k}$) of the vertices of A, are adjacent to at least $\frac{\delta_1^2 n}{4k}$ vertices in each of B

and B'. Let a be one of these normal vertices, and set $B_0 = B \cap N(a)$ and $B'_0 = B' \cap N(a)$. Similarly, almost all of the vertices of A', are adjacent to at least $\frac{\delta_1^3 n}{8k}$ vertices in each of B_0 and B'_0. Let a' be one of these normal vertices and let b and b' be any vertices in B_0 and B'_0, respectively. In extending L to L' we may have violated condition (3) for the first rung $u'v'$ by using up some of its neighbors. So now, in a similar way, we choose $a'' \in U_i^* \setminus L'$ and $b'' \in U_i^* \setminus L'$ such that $u' \sim b'' \sim a'' \sim v'$ and $\deg(a'', V_i^* \setminus L')$, $\deg(b'', U_i^* \setminus L') \geq \frac{\delta_1 n}{2k} + 1$. We then add $a''b''$ to L' as a first rung to obtain L'' satisfying (3) Continuing in this fashion we finally obtain the desired ladder L^i.

For each $i \in [r]$, set $U_i^1 = U_i^* \setminus L^i$ and $V_i^1 = V_i^* \setminus L^i$. Then

$$\left|U_i^1\right|, \left|V_i^1\right| \geq \left(1 - \varepsilon_1 - 32\sqrt{\delta_1}\right) \frac{n}{k} \geq (1 - \delta_2) \frac{n}{k}.$$

By the slicing lemma each of the pairs (U_i^1, V_i^1) and $\left(V_i^1, U_{i+1}^1\right)$ are $2\varepsilon_1$-regular with density at least $\delta_1 - \varepsilon_1$. However they may not be super-regular. Our next task is to reassign some vertices so that each of the pairs (U_i^1, V_i^1) is (ε, δ)-super-regular. By Lemma 6 there are at most $\varepsilon_1 \frac{n}{k}$ vertices $u \in U_i^1$ with $\deg(u, V_i^1) \leq (\delta_1 - \varepsilon_1)|U_i^1|$. Similarly, there are at most $\varepsilon_1 \frac{n}{k}$ vertices $v \in V_i^1$ with $\deg(v, U_i^1) \leq (\delta_1 - \varepsilon_1)|V_i^1|$. Move these to the (currently empty) exceptional clusters U_0 and V_0. So $|U_0|, |V_0| \leq \varepsilon_1 n$. We shall redistribute these exceptional vertices to get new clusters U_i^2 and V_i^2, for all $i \in [r]$. Using Claim 3 with $\sigma_1 = \varepsilon_1$ and $\sigma_2 = \delta_1$ we can do this so that

(1) If v is reassigned to V_i^2 then $\deg(v, U_i^1) \geq \delta_1 \frac{n}{k}$ and if u is reassigned to U_i^2 then $\deg(u, V_i^1) \geq \delta_1 \frac{n}{k}$.
(2) For every $i \in [r]$, $|V_i^2| - |U_i^2| = |V_i^1| - |U_i^1|$.
(3) For each $i \in [r]$, at most $\sqrt{\varepsilon_1} \frac{n}{k}$ vertices were reassigned to V_i^2 (U_i^2).
(4) For each $i \in [r]$, at most $\sqrt{\varepsilon_1} \frac{n}{k}$ vertices were reassigned from V_i^2 (U_i^2).

Recall that $\delta = \frac{\delta_1}{2}$ and $\varepsilon = 6\varepsilon_1^{1/4}$. Now the minimum degree of each pair (V_i^2, U_i^2) is at least

$$(\delta_1 - \varepsilon_1 - \sqrt{\varepsilon_1}) \frac{n}{k} \geq (\delta_1 - 3\sqrt{\varepsilon_1}) \left|U_i^2\right| \geq \delta \left|U_i^2\right|.$$

By Lemma 8 each pair (U_i^2, V_i^2) is ε-regular. Thus each pair (U_i^2, V_i^2) is (ε, δ)-super-regular. Similarly, each pair $\left(V_i^2, U_{i+1}^2\right)$ is ε-regular with density at least δ. For each $i \in [r-1]$ choose $v_i \in V_i^2$ such that $\deg\left(v_i, U_{i+1}^2\right) \geq \frac{\delta n}{2k}$ and let $A_{i+1} = U_{i+1}^2 \cap N(v_i)$. Using Lemma 6, choose $u_{i+1} \in A_{i+1}$ such that $\deg(u_{i+1}, V_i^2) \geq \frac{\delta n}{2k}$. Let $U_i^3 = U_i^2 \setminus \{u_i\}$ and $V_i^3 = V_i^2 \setminus \{v_i\}$, where $\{u_1\} = \emptyset = \{v_r\}$. Then (U_i^3, V_i^3) is still an (ε, δ)-super-regular pair and $|U_i^3| = |V_i^3|$, for all $i \in [r]$. Let $B_i^1 = V_i^3 \cap N(u_{i+1})$, $A_{i+1}^0 = U_{i+1}^3 \cap N(v_i)$, $A_i^1 = U_i^3 \cap N(v_i)$, and $B_{i+1}^0 = V_{i+1}^3 \cap N(u_{i+1})$. Let $x_i y_i$ be the first rung of L^i, where $x_i \in U$ and $y_i \in V$, and let $w_i z_i$ be the last rung of L^i, where $w_i \in U$ and $z_i \in V$. Finally let $X_i = U_i^3 \cap N(y_i)$, $Y_i = V_i^3 \cap N(x_i)$, $W_i = U_i^3 \cap N(z_i)$, and $Z_i = V_i^3 \cap N(w_i)$. Note that each of X_i^3, Y_i^3, W_i^3, and Z_i^3 has size at least $\frac{\delta n}{2k} \geq \beta \frac{n}{k}$.

We now apply the Blow-Up Lemma to each pair (U_i^3, V_i^3) to find a spanning ladder K^i whose first rung is contained in $A_i^0 \times B_i^0$, whose second rung is contained in $X_i \times Y_i$, whose third rung is contained in $W_i \times Z_i$, and whose last rung is contained in $A_i^1 \times B_i^1$. This is possible since $\gamma \frac{n}{2k} > 8$. Clearly we can insert L^i between the second and third rungs of K^i to obtain a ladder M^i spanning $U_i^3 \cup V_i^3$. Finally, $M^1 v_1 u_2 M^2 \ldots v_{r-1} u_r M^r$ is a spanning ladder of G.

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