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# Periodic Solutions in Periodic Delayed Gause-Type Predator-Prey Systems

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**Abstract**

Reasonable sufficient conditions are obtained for the existence of positive periodic solutions in periodic delayed Gause-type predator-prey systems. Our approach involves the application of coincidence degree theorem and estimations of uniform upper bounds on solutions. This method imposes minimum restrictions on the form and magnitude of time delays. Indeed, our results are applicable to discrete, distributed and state-dependent delays. Our results indicate that seasonal effects on population models often lead to synchronous solutions. In addition, we may conclude that when both seasonality and time delay are present, the seasonality is often the generating force for the often observed fluctuations in population densities, including the inherently oscillatory predator-prey dynamics.

**Keywords:** Coincidence degree, Periodic solution, delay equation, state-dependent delay, predator-prey model.
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# 1 Introduction

It is often observed that populations in the real world tend to fluctuate. This is especially so for predotor-prey interactions. There are three typical approaches for modeling such behavior: (i)introduce more species into the model, and consider the higher dimensional systems (like predator-prey interactions, May(1974)); (ii)assume that the per capita growth function is time dependent and periodic in time; (iii)take into account the time delay effect in the population dynamics(Smith and Kuang(1992), Zhao et al.(1997)). Although all of them are good mechanisms of generating periodic solutions(and therefore offer some explanations to the often observed oscillatory behavior in population densities), it does not give us any insight as which is the real generating or dominating force behind the oscillatory behavior if only one of such mechanism is considered. Naturally, more realistic and interesting models of population interactions should take into account both the seasonality of the changing environment and the effects of time delays. Therefore, it is interesting and important to study the following general nonlinear nonautonomous delayed Gause-type predator-prey system:

$$\begin{cases} \dot{x}(t) = x(t)[g(t,x(t)) - p(x(t))y(t)], \\ \dot{y}(t) = y(t)\left[-\alpha + h\left(\int_{-\gamma}^{-\gamma_0} x(t+\theta)\,d\eta(\theta)\right)\right], \end{cases} \tag{1.1}$$

where

(A1) $p \in C^1([0,+\infty), R)$, and $p(x) > 0$ for $x \geq 0$;

(A2) $h(x) \in C^1([0,+\infty), R)$ and $h'(x) > 0$ for $x \geq 0$; $h(0) = 0$ and there exists $x_0 > 0$ such that $h(x_0) = \alpha$;

(A3) $g(t,x) \in C^1([0,+\infty) \times R, R)$, $g(t+\omega, x) = g(t,x)$ for $(t,x) \in R \times [0,+\infty)$, where $\omega$ is a nonnegative constant. There exists a constant $H > 0$ such that $g(t,x) \leq H$ for $(t,x) \in R \times [0,+\infty)$. In addition, $\int_0^\omega g(t,x_0)\,dt > 0$.

(A4) $r_0, r > 0$ are constants, $\eta$ is a nondecreasing function satisfying $\eta(-\gamma_0^+) - \eta(-\gamma^-) = 1$.

Readers familiar with predator-prey models may notice that the above assumptions are needed to make the system a standard Gause-type predator-prey model and to ensure the existence of a positive steady state when the system is reduced to an autonomous ODE model.

At the end of this paper, we will see that our results for the above system can be easily extended to the one with a state-dependent delay.

Existing results on the existence of periodic solutions in periodic systems (population models, in particular) often fall into one of these three categories. (1): the results of the applications of contraction principle or fluctuation principle, which establish both the existence and attractivity of the periodic solutions in periodic equations with time delay(Kuang(1993), p181); (2) the existence simply follow the observation that the periodic solution exist when there is no time delay and this periodic solution remains so when time delay is a multiple of the period of the periodic equation(Gopalsamy et al.(1990), Zhang and Gopalsamy(1990)); (3) the results of the application of Horn's asymptotic fixed point theorem(Freedman and Wu(1992), Tang and Kuang(1997)). While these methods often allow the investigator to address the stability issues of the periodic solutions, the conditions for the existence part are often unnecessary numerous, tedious, stringent, and difficult to satisfy. Specifically, all the above methods are ill suited to problems with state-dependent delay equations.

By employing the powerful and effective coincidence degree method, we found that the existence of periodic solutions in periodic predator-prey systems with or without state-dependent delay require only a set of natural and easily verifiable conditions. These conditions are readily satisfied in many realistic population models. This strongly suggests that seasonal effects on population models indeed often lead to synchronous solutions. In addition, we may conclude that when both seasonality and time delay are present and deserve consideration, the seasonality is often the generating force for the often observed oscillatory behavior in population densities.

## 2 Main results

The method to be used in this paper involves the application of the continuous theorem of coincidence degree(Gaines and Mawhin(1977), p.40). This requires us to introduce a few notations.

Let $X, Y$ be real Banach spaces, $L : \text{dom } L \subset X \to Y$ a Fredholm mapping of index zero (Index $L = \dim \ker L - \text{codim Im } L$) and $P : X \to X$, $Q : Y \to Y$ are continuous projectors such that $\text{Im } P = \text{Ker } L$, $\text{Ker } Q = \text{Im } L$ and $X = \text{Ker } L \oplus \text{Ker } P$, $Y = \text{Im } L \oplus \text{Im } Q$. Consequently, the restriction $Lp$ of $L$ to $\text{dom } L \cap \text{Ker } P$ is one-to-one and onto map to $\text{Im } L$, so that its (algebraic) inverse $K_p : \text{Im } L \to \text{dom } L \cap \text{Ker } P$ is well defined. Denote by $J : \text{Im } Q \to \text{Ker } L$ an isomorphism of $\text{Im } Q$ onto $\text{Ker } L$.For convenience we cite the continuous theorem from Gaines and Mawhin((1977), p.40):

**Theorem A** *Let $\Omega \subset X$ be an open bounded set and $N : X \to Y$ be a continuous operator which is L-compact on $\bar{\Omega}$ (i.e., $QN : \bar{\Omega} \to Y$ and $K_p(I-Q)N : \bar{\Omega} \to X$ are compact). Assume*

- *(a) for each $\lambda \in (0,1)$, every solution $x$ of $Lx = \lambda Nx$ is such that $x \notin \partial\Omega$,*
- *(b) $QNx \neq 0$ for each $x \in$ Ker $L \cap \partial\Omega$,*
- *(c) the Brouwer degree deg $[JQN, \Omega \cap \text{Ker } L, 0] \neq 0$.*

*Then the operator equation $Lx = Nx$ has at least one solution in* dom $L \cap \bar{\Omega}$.

Consider the following system

$$\begin{cases} \dot{u}(t) = g(t, e^{u(t)}) - p(e^{u(t)})e^{v(t)}, \\ \dot{v}(t) = -\alpha + h\left(\int_{-\gamma}^{-\gamma_0} e^{u(t+\theta)}\, d\eta(\theta)\right). \end{cases} \tag{2.1}$$

It is easy to see that if system (2.1) has an $\omega$-periodic solution $(u^*(t), v^*(t))$, then $(e^{u^*(t)}, e^{v^*(t)})$ is a positive $\omega$-periodic solution of system (1.1).

In order to use continuation theorem of coincidence degree theory to establish the existence of an $\omega$-periodic solution of system (2.1), we take $X = \{(u,v)^T \in C(R, R^2) : u(t+\omega) = u(t), v(t+\omega) = v(t)\}$ and

$$\|(u,v)^T\| = \max_{t\in[0,\omega]} |u(t)| + \max_{t\in[0,\omega]} |v(t)|.$$

With this norm, $X$ is a Banach space. Let

$$L : \text{dom } L \to X, \quad (u,v)^T \mapsto (\dot{u}(t), \dot{v}(t))^T,$$

where dom $L = \{(u,v)^T \in X : (u,v)^T \in C^1(R,R^2)\}$. Obviously, Ker $L = \{(u,v)^T \in X : (u,v)^T = (c_1,c_2)^T \in R^2\}$, Im $L = \{(u,v)^T \in X : \int_0^\omega u(t)\,dt = \int_0^\omega v(t)\,dt = 0\}$ is closed in $X$ and dim Ker $L =$ codim Im $L = 2$. Therefore, $L$ is a Fredholm mapping of index 0.

The following is our main result.

**Theorem 2.1** *Suppose that* (A1)–(A4) *hold, then system (1.1) has a positive $\omega$-periodic solution.*

**Proof.** To make use of Theorem A, we need to define a continuous operator $N$,

$$N : X \to X, \quad \begin{bmatrix} u \\ v \end{bmatrix} \mapsto \begin{bmatrix} g(t, e^{u(t)}) - p(e^{u(t)})e^{v(t)} \\ -\alpha + h\left(\int_{-\gamma}^{-\gamma_0} e^{u(t+\theta)}\right)\,d\eta(\theta) \end{bmatrix}.$$

We define the two continuous projectors as follows:

$$\begin{aligned} P &: X \to X, \quad P((u,v)^T) = \left(\frac{1}{\omega}\int_0^\omega u(t)\,dt, \frac{1}{\omega}\int_0^\omega u(t)\,dt\right)^T, \\ Q &: X \to X, \quad Q((u,v)^T) = \left(\frac{1}{\omega}\int_0^\omega u(t)\,dt, \frac{1}{\omega}\int_0^\omega v(t)\,dt\right)^T. \end{aligned}$$

Then, through an elementary computation, we can obtain the explicit form of the corresponding inverse $K_p$ : Im $L \to$ Ker $P \cap$ dom $L$ of $L_p$ as:

$$\begin{bmatrix} u \\ v \end{bmatrix} \mapsto \begin{bmatrix} \int_0^t u(s)\,ds - \frac{1}{\omega}\int_0^\omega \int_0^h u(s)\,ds\,dh \\ \int_0^t v(s)\,ds - \frac{1}{\omega}\int_0^\omega \int_0^h v(s)\,ds\,dh \end{bmatrix}.$$

Notice that $QN : X \to X$ takes the form of

$$\begin{bmatrix} u \\ v \end{bmatrix} \mapsto \begin{bmatrix} \frac{1}{\omega}\int_0^\omega [g(t, e^{u(t)}) - p(e^{u(t)})e^{v(t)}]\,dt \\ \frac{1}{\omega}\int_0^\omega \left[-\alpha + h\left(\int_{-\gamma}^{-\gamma_0} e^{u(t+\theta)}\,d\eta(\theta)\right)\right]\,dt \end{bmatrix}.$$

Hence, we have $K_p(1-Q)N : X \to X$ takes the form of

$$\begin{aligned} \begin{bmatrix} u \\ v \end{bmatrix} \mapsto & \begin{bmatrix} \int_0^t [g(s, e^{u(s)}) - p(e^{u(s)})e^{v(s)}]\,ds \\ \int_0^t \left[-\alpha + h\left(\int_{-\gamma}^{\gamma_0} e^{u(s+\theta)}\,d\eta(\theta)\right)\right]\,ds \end{bmatrix} \\ & - \begin{bmatrix} \frac{1}{\omega}\int_0^\omega \int_0^h [\rho(s, e^{u(s)}) - p(e^{u(s)})e^{v(s)}]\,ds \\ \frac{1}{\omega}\int_0^\omega \int_0^h \left[-\alpha + h\left(\int_{-\gamma}^{\gamma_0} e^{u(s+\theta)}\,d\eta(\theta)\right)\right]\,ds\,dh \end{bmatrix} \\ & + \begin{bmatrix} \left(\frac{1}{2} - \frac{t}{\omega}\right)\int_0^\omega [g(s, e^{u(s)}) - p(e^{u(s)})e^{v(s)}]\,ds \\ \left(\frac{1}{2} - \frac{t}{\omega}\right)\int_0^\omega \left[-\alpha + h\left(\int_{-\gamma}^{-\gamma_0} e^{u(s+\theta)}\,d\eta(\theta)\right)\right]\,ds \end{bmatrix}. \end{aligned}$$

Clearly, $QN$ and $K_p(1-Q)N$ are continuous by the Lebesgue theorem and, furthermore, $QN(\overline{\Omega})$ and $K_p(1-Q)N(\overline{\Omega})$ are relatively compact for any open bounded set $\Omega \subset X$. Hence,

$N$ is $L$-compact on $\overline{\Omega}$ for any open bounded set $\Omega \subset X$. The corresponding operator equation $Lx = \lambda Nx$, $\lambda \in (0,1)$, takes the form of

$$\begin{cases} \dot{u}(t) = \lambda[g(t, e^{u(t)}) - p(e^{u(t)})e^{v(t)}], \\ \dot{v}(t) = \lambda\left[-\alpha + h\left(\int_{-\gamma}^{-\gamma_0} e^{u(t+\theta)}\, d\eta(\theta)\right)\right], \end{cases} \qquad \lambda \in (0,1). \tag{2.2}$$

Suppose that $(u(t), v(t))^T \in X$ is a solution of system (2.2) for a certain $\lambda \in (0,1)$. Integrating system (2.2) over the interval $[0,\omega]$ we obtain

$$\int_0^\omega g(t, e^{u(t)})\, dt = \int_0^\omega p(e^{u(t)})e^{v(t)}\, dt \tag{2.3}$$

and

$$\int_0^\omega h\Big(\int_{-\gamma}^{-\gamma_0} e^{u(t+\theta)}\, d\eta(\theta)\Big)\, dt = \alpha\omega. \tag{2.4}$$

It follows from (A3) and (2.3) that

$$\int_0^\omega p(e^{u(t)})e^{v(t)}\, dt \le H\omega. \tag{2.5}$$

Denote $\Delta_1 = \{t \in [0,\omega] : g(t, e^{u(t)}) \ge 0\}$, $\Delta_2 = \{t \in [0,\omega] : g(t, e^{u(t)}) < 0\}$, then

$$\int_0^\omega g(t, e^{u(t)})\, dt = \int_{\Delta_1} g(t, e^{u(t)})\, dt + \int_{\Delta_2} g(t, e^{u(t)})\, dt.$$

From this, (A3) and (2.3), we find

$$\begin{aligned} -\int_{\Delta_2} g(t, e^{u(t)})\, dt &= \int_{\Delta_1} g(t, e^{u(t)})\, dt - \int_0^\omega p(e^{u(t)})e^{v(t)}\, dt \\ &< \int_{\Delta_1} g(t, e^{u(t)})\, dt \le H\omega. \end{aligned}$$

Hence

$$\begin{aligned} \int_0^\omega |g(t, e^{u(t)})|\, dt &= \int_{\Delta_1} |g(t, e^{u(t)})|\, dt + \int_{\Delta_2} |g(t, e^{u(t)})|\, dt \\ &= \int_{\Delta_1} g(t, e^{u(t)})\, dt - \int_{\Delta_2} g(t, e^{u(t)})\, dt 2H\omega. \end{aligned} \tag{2.6}$$

By (2.2), (2.5) and (2.6), we obtain

$$\begin{aligned} \int_0^\omega |\dot{u}(t)|\, dt &\le \int_0^\omega |g(t, e^{u(t)}) - p(e^{u(t)})e^{v(t)}|\, dt \\ &< \int_0^\omega |g(t, e^{u(t)})|\, dt + \int_0^\omega p(e^{u(t)})e^{v(t)}\, dt \\ &< 3H\omega. \end{aligned} \tag{2.7}$$

Moreover, from (2.2) and (2.4) it follows that

$$\begin{aligned}\int_0^\omega |\dot{v}(t)|\,dt &\le \int_0^\omega \Big|-\alpha+h\Big(\int_{-\gamma}^{-\gamma_0} e^{u(t+\theta)}\,d\eta(\theta)\Big)\Big|\,dt\\ &\le \int_0^\omega \alpha\,dt+\int_0^\omega h\Big(\int_{-\gamma}^{-\gamma_0} e^{u(t+\theta)}\,d\eta(\theta)\Big)\,dt\\ &= 2\alpha\omega. \qquad (2.8)\end{aligned}$$

From (A3) and (2.4), it is easy to see that there exists a constant $B_1 > 0$ and a point $t_1 \in [0,\omega]$ such that

$$|u(t_1)| < B_1.$$

It follows from this and (2.7) that

$$\begin{aligned}|u(t)| &\le |u(t_1)|+\int_0^\omega |\dot{u}(t)|\,dt\\ &< B_1+3H\omega \overset{\text{def}}{=} B_2. \qquad (2.9)\end{aligned}$$

In view of (A1), (A2), (2.3), (2.5) and (2.9), we have

$$p(e^{B_2})\int_0^\omega e^{v(t)}\,dt \le \int_0^\omega p(e^{u(t)})e^{v(t)}\,dt \le H\omega \qquad (2.10)$$

and

$$\begin{aligned}p(0)\int_0^\omega e^{v(t)}\,dt &\ge \int_0^\omega p(e^{u(t)})e^{v(t)}\,dt\\ &= \int_0^\omega \rho(t,e^{u(t)})\,dt \ge \underline{g}\omega, \qquad (2.11)\end{aligned}$$

where $\underline{g} = \min_{\substack{t\in[0,\omega]\\ u\in[-B_2,B_2]}} \{g(t,e^u)\}$.

By (2.10), (2.11) and the continuity of $v(t)$, one can easily see that there exists a constant $B_3$ and a point $t_2 \in [0,\omega]$ such that

$$|v(t_2)| < B_3.$$

From this and (2.8) we obtain that

$$\begin{aligned}|v(t)| &\le |v(t_2)|+\int_0^\omega |\dot{v}(t)|\,dt\\ &< 2\alpha\omega+B_3 \overset{\text{def}}{=} B_4.\end{aligned}$$

Denote $B = B_2+B_4+\ln x_0+\ln(\omega^{-1}\int_0^\omega g(t,x_0)\,dt\,p(xb))$ and take $\Omega = \{(u(t),v(t))^T \in X : \|(u,v)^T\| < B\}$. It is clear that $\Omega$ verifies the requirement (a) in Lemma A.

When $(u,v) \in \partial\Omega\cap$ Ker $L = \partial\Omega\cap R^2$, $(u,v)$ is a constant vector in $R^2$ with $|u|+|v| = B$. Then

$$QN\begin{bmatrix} u\\ v\end{bmatrix} = \begin{bmatrix} \frac{1}{\omega}\int_0^\omega g(t,e^u)\,dt - p(e^u)e^v\\ -\alpha+h(e^u)\end{bmatrix} \ne 0.$$

Moreover, take $J = I : \text{ Im } Q \rightarrow \text{ Ker } L$, $x \mapsto x$. Then in view of (A2), it is easy to see that

$$\begin{aligned}&\deg\,\{JQN((u,v)^T), \Omega \cap \text{ Ker } L, 0\}\\ &= \deg\,\{QN((u,v)^T), \Omega \cap R^2, 0\}\\ &= \text{sgn}\,\{h'(x_0)p(x_0)x_0e^{\bar{v}}\} = 1 \neq 0,\end{aligned}$$

where $e^{\bar{v}} = \omega^{-1}\int_0^\omega \rho(t, x_0)\,dt/p(x_0)$.

By now we have proved that $\Omega$ verifies all the requirements of Lemma A and hence system (2.1) has an $\omega$-periodic solution. The proof is complete.

By Theorem 2.1, one can easily see that

**Corollary 1** *Suppose that* (A1)–(A3) *hold. Then the following system*

$$\begin{cases} \dot{x}(t) = x(t)[y(t, x(t)) - p(x(t))y(t)],\\ \dot{y}(t) = y(t)[-\alpha + h(x(t-\tau))] \end{cases}$$

*has a positive $\omega$-periodic solution, where $\tau > 0$ is a constant.*

Next consider the following infinite delayed Gause-type prey-predator systems

$$\begin{cases} \dot{x}(t) = x(t)[g(t, x(t)) - p(x(t))y(t)],\\ \dot{y}(t) = y(t)\left[-\alpha + h\left(\int_{-\infty}^0 x(t+\theta)\,d\eta(\theta)\right)\right], \end{cases} \tag{2.12}$$

where $\eta$ is a nondecreasing function satisfying $\eta(0^+) - \eta(-\infty) = 1$.

**Theorem 2.2** *Suppose that* (A1)–(A3) *hold. Then system (2.12) has a positive $\omega$-periodic solution.*

**Proof.** The proof is similar to the proof of Theorem 2.1 and hence is omitted here.

Finally, consider the following state dependent delay Gause-type prey-predator system.

$$\begin{cases} \dot{x}(t) = x(t)[g(t, x(t)) - p(x(t))y(t)],\\ \dot{y}(t) = y(t)[-\alpha + h(x(t - \tau(x(t))))] \end{cases} \tag{2.13}$$

where $\tau \in C(R, R^+)$.

**Theorem 2.3** *Suppose that* (A1)–(A3) *hold. Then the system has a positive $\omega$-periodic solution.*

**Proof.** Again, the proof is similar to the proof of Theorem 2.1 and is thus omitted.

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