Lecture 6, Th., Sept. 6. Hw 2 is due on Th., Sept. 27

Reading homework: Chapter 2

Nonlinear Difference Equations

1. Stability of first order nonlinear difference equations. We consider first the scalar equation

$$x_{n+1} = f(x_n), \qquad\qquad f(x) \in C^1. \tag{1.1}$$

We say $\overline{x}$ is a steady state solution (equilibrium) of (1.1) if $\overline{x} = f(\overline{x})$.

DEFINITION 1. *The steady state solution $\overline{x}$ is stable if for any positive constant ε, there is a δ such that $|x_0 - \overline{x}| < \delta$ implies that for all $n > 0$, $|x_n - \overline{x}| < \varepsilon$. If in addition, $\lim_{n\to\infty} x_n = \overline{x}$, then we say that the steady state solution $\overline{x}$ is asymptotically stable.*

Notice that in the textbook, stable is actually referred as asymptotically stable. The following simple theorem is very useful. We provide its rigorous proof.

THEOREM 1. *The steady state solution $\overline{x}$ of (1.1) is asymptotically stable if $|df(\overline{x})/dx| < 1$.*

Proof. Since $f(x) \in C^1$ and $|df(\overline{x})/dx| < 1$, there is a $\varepsilon_1 > 0$ such that $|x_0 - \overline{x}| \le \varepsilon_1$ ensures that $|df(x_0)/dx| < 1$. Then

$$\lambda \equiv \max\{|df(x_0)/dx| : |x_0 - \overline{x}| \le \varepsilon_1\} < 1.$$

Given $\varepsilon > 0$, let $\delta = \min\{\varepsilon/2, \varepsilon_1/2\}$. Recall that by the mean value theorem, we have that

$$f(x_0) = f(\overline{x}) + f'(\xi)(x_0 - \overline{x})$$

for some ξ in between x_0 and $\overline{x}$. Since $\overline{x} = f(\overline{x})$, we have

$$|x_1 - \overline{x})| = |f(x_0) - \overline{x})| = |f'(\xi)(x_0 - \overline{x})| \le \lambda |x_0 - \overline{x}| < \delta.$$

Continue this way, we obtain that

$$|x_n - \overline{x})| \le \lambda^n |x_0 - \overline{x}| < \delta.$$

Clearly, $\lim_{n\to\infty} x_n = \overline{x}$. ∎

A simple application of this theorem to the discrete logistic equation (see example 2 on page 44) $x_{n+1} = rx_n(1 - x_n)$ yields that $\overline{x} = 1 - 1/r$ exists and is asymptotically stable if $1 < r < 3$.

2. Some general global stability results. It is easy to see from intuition or examples that in general, local asymptotical stability of a steady state does not imply that this steady state is also global asymptotical stability with respect to solutions of interest. In applications, many interesting models are in the form of nonlinear difference equations or systems. Obviously, in these cases, linear stability alone does not give enough information about the steady state or the system.

In this section we shall present some general yet effective results on global asymptotical stability of certain scalar nonlinear difference equations. Specifically, we are concerned here

$$x_{n+1} = f(x_n), \qquad x_0 = a, \tag{2.1}$$

where $f(x)$ is continuous and monotone or has a single peak(hump) in the domain of interest. Such functions are often encountered in applications. Some of the results of this section will be applied to specific equations in subsequent sections. We shall assume that the equation is defined on an interval of the real line. Without lose of generality, we shall further assume that such interval is invariant for the considered equation. Such assumptions are often readily met in applications. In what follows, we denote by x_n or $x_n(a)$ the solution of (2.1).

THEOREM 2.1. *Assume that I is an interval of real line and that I is invariant with respect to* (2.1). *Assume further that* (2.1) *has a steady state $x^* \in I$ and that $|(f(x)-x^*)/(x-x^*)| \leq \alpha < 1, x \in I$, where α is a positive constant. Then this steady state x^* is unique and is globally asymptotically stable on I.*

Proof. Let $x_0 \in I$. Since I is invariant, we have $x_n \in I$ for all positive integer n. It is easy to see that

$$\frac{|x_{n+1} - x^*|}{|x_n - x^*|} = \frac{|f(x_n) - f(x^*)|}{|x_n - x^*|} \leq \alpha.$$

We thus see that $|(x_{n+1} - x^*)/(x_n - x^*)| \leq \alpha$. This clearly indicates that $|x_n - x^*| \leq \alpha^n |x_0 - x^*|$. Hence $\lim_{n\to\infty} x_n = x^*$. This proves the theorem. ∎

COROLLARY 2.2. *Assume that $|f'(x)| \leq \alpha < 1, x \in I$, where α is a positive constant and I is an interval of real line. Assume further that I is invariant with respect to* (2.1). *If* (2.1) *has a steady state $x^* \in I$. Then this steady state is unique and it is globally asymptotically stable on I.*

Proof. The proof is straightforward by noticing that

$$\frac{x_{n+1} - x^*}{x_n - x^*} = \frac{f(x_n) - f(x^*)}{x_n - x^*} = f'(z),$$

where z is in between x_n and x^*. ∎