

Lectures 19, Tu., October 23, 2007

Reading homework: Chapter 6

1. Lotka-Volterra predator-prey model. Most of the interesting dynamics in biosciences are related to interactions among species. One of the first models to incorporate interactions between predators and prey was proposed in 1925 by the American biophysicist Alfred Lotka and the Italian mathematician Vito Volterra in 1926. It takes the form

$$\begin{cases} \frac{dx}{dt} = x(a - by), & x(0) > 0, \\ \frac{dy}{dt} = -y(c - ex), & y(0) > 0, \end{cases} \quad (1.1)$$

where y is the number or density of the predator, x is number or density of its prey. a is the natural prey growth rate in the absence of predation, c is the predator's natural death rate in the absence of food, b is the prey loss rate per encounter of a unit of prey to a unit of predator, e is the efficiency of turning predated prey biomass into predator's biomass.

We have shown that solution of (1.1) is positive and bounded. Indeed, we have shown that its solution is periodic. However, we present below only the result on positivity of the solution. The existence and uniqueness can be established by a basic theorem of ODE. It can also be established by solving it directly to obtain an implicit but closed form solution. The method used in the following theorem in addressing the positivity of solutions is applicable in many other situations.

THEOREM 1. *The solution of (1.1) is positive.*

Proof. We show first that the solution of (1.1) is positive. If not, then there is a $t_1 > 0$ such that the component x and y of the solution are positive for $t \in [0, t_1]$ but $x(t_1)y(t_1) = 0$. Assume first that $y(t_1) = 0$. Then for $t \in [0, t_1]$, $y'(t) > -cy(t)$ which yields $y(t) \geq y(0)e^t$ for $t \in [0, t_1]$. In particular, we have $y(t_1) > 0$, a contradiction. Assume now that $x(t_1) = 0$. Let $y_m = \max\{y(t) : t \in [0, t_1]\}$. Then for $t \in [0, t_1]$, we have $x'(t) \geq (a - by_m)x$ which implies that $x(t_1) \geq x(0)e^{(a-by_m)t_1} > 0$. This is a contradiction and the positivity of the solution is established. ■

Since

$$\frac{dy}{dx} = \frac{y(ex - c)}{x(a - by)}.$$

We see that

$$\frac{(a - by)dy}{y} = \frac{(ex - c)dx}{x}$$

A straightforward integration yield that

$$by - a \ln y + ex - c \ln x = C_0 \equiv by(0) - a \ln y(0) + ex(0) - c \ln x(0).$$

Since both $by - a \ln y$ and $ex - c \ln x$ are bounded from below, we see that both x and y must be bounded and bounded away from zero.

The Lotka-Volterra model is one of the earliest predator-prey models. It forms the basis of many models used today in the analysis of population dynamics. Unfortunately, in its original form Lotka-Volterra has some significant problems. The

most noteworthy is that the positive equilibrium is neutrally stable. The predator and prey populations cycle endlessly without settling down. While this cycling has been observed occasionally in nature, it is far from common and when it appears the limit cycle is usually does not dependent on initial data. Hence Lotka-Volterra predator-prey model by itself may not be appropriate to model any specific predator-prey systems. Additional context specific structures must be incorporated to make a predator-prey model more relevant to nature.