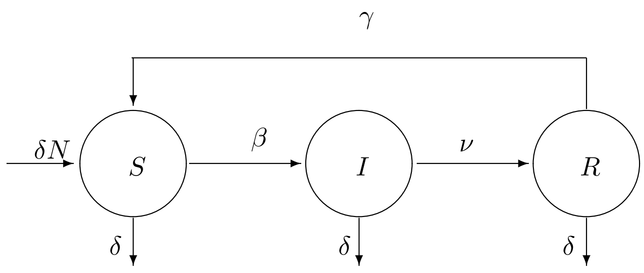


FIG. 1.1. *Flow diagram for the modified Kermack-McKendrick model with loss of immunity and vital dynamics*

# Lectures 26, Th., Nov. 16

**Reading homework: Chapter 7 of reference 1**

**1. The Kermack-McKendrick model with vital dynamics.** The Kermack-McKendrick model assumes a completely homogeneous population with no age, spatial, birth, death or social structure. It is a minimum model with many directions for improvement. In this lecture, we include in the Kermack-McKendrick model the simplest form of birth and death processes. **This is essentially answers the exercise 26 on page 263.** This modified Kermack-McKendrick model takes the form of,

$$S' = -\beta SI + \gamma R + \delta N - \delta S, \quad I' = \beta SI - (\nu + \delta)I, \quad R' = \nu I - (\gamma + \delta)R. \tag{1.1}$$

We assume throughout the rest of this lecture that $S(0) > 0$ and $I(0) > 0$. Here $\beta$ is the infection rate, and $\gamma$ is the recovery rate. The Kermack-McKendrick model was brought back to life by Anderson and May (1979). More complicated versions of the Kermack-McKendrick model that better reflect the actual biology of a given disease are often used.

The key value governing the time evolution of these equations is the so-called epidemiological threshold,

$$R_0 = \beta S(0)/(\nu + \delta). \tag{1.2}$$

$R_0$ is defined as the number of secondary infections caused by a single primary infection. Which is equivalent to say that $R_0$ is the number of infections produced by a single infected individual in its infectious period. It determines the number of people infected by contact with a single infected person before his death or recovery. This number is also referred in literature as basic reproduction (reproductive) ratio, or basic reproduction (reproductive) number. When $R_0 < 1$, each person who contracts the disease will infect fewer than one person before dying or recovering, so the outbreak will die out ($dI/dt < 0$). When $R_0 > 1$, each person who gets the disease will infect more than one person, so the epidemic will spread ($dI/dt > 0$). $R_0$ is probably the single most important quantity in epidemiology. Note that the result $R_0$ is dependent on the expression of $dI/dt$.

Clearly, the total population stays constant ($N = S + I + R = N(0)$). We show first that solutions of (1.1) are positive for $t > 0$.

THEOREM 1.1. *Solutions of (1.1) are positive for $t > 0$.*

**Proof.** We show first that $S$ can not be the first population to go extinct. If not, then there is a $t_1 > 0$, such that $S(t), I(t).R(t) > 0$ for $t \in [0, t_1)$, $S(t_1) = 0$ and $S'(t_1) \leq 0$. However, from the model, we have

$$S'(t_1) = \gamma R + \delta N(0) > 0.$$

This is a contradiction.

Now we shall show that $I$ also can not be the first population to go extinct. If not, then there is a $t_1 > 0$, such that $S(t), I(t).R(t) > 0$ for $t \in [0, t_1)$, $I(t_1) = 0$ and $I'(t_1) \leq 0$. Let

$$m = \min\{\beta S - (\nu + \delta) : t \in [0, t_1]\} > -\infty.$$

Then we have

$$I'(t) \geq mI(t)$$

which implies that for $t \in [0, t_1]$,

$$I(t) \geq I(0)e^{mt}.$$

In particular, we have $I(t_1) \geq I(0)e^{mt_1} > 0$, a contradiction.

Finally, we shall show that $R$ can not be the first population to go extinct. If not, then there is a $t_1 > 0$, such that $S(t), I(t).R(t) > 0$ for $t \in [0, t_1)$, $I(t_1) > 0$, $R(t_1) = 0$ and $R'(t_1) \leq 0$. However, from the model, we have

$$R'(t_1) = \nu I(t_1) > 0.$$

This is a contradiction. ∎

It is easy to show an endemic steady state $E^* = (S^*, I^*, R^*)$ exists iff that $R_0 > 1$. When it exists, we have

$$S^* = \frac{\nu + \delta}{\beta}, \ I^* = \frac{(\nu + \delta)(N(0) - S^*)}{\nu + \delta + \gamma}, \ R^* = \frac{\nu I^*}{\gamma + \delta}.$$

Since $N(t) \equiv N(0)$, we can simply study the dynamics of the reduced $SI$-system.

$$S' = -\beta SI + \gamma(N - S - I) + \delta N - \delta S, \quad I' = \beta SI - (\nu + \delta)I. \tag{1.3}$$

The Jacobian at $(S^*, I^*)$ is

$$J(S^*, I^*) = \begin{pmatrix} -\beta I^* - \gamma - \delta & -\beta S^* - \gamma \\ \beta I^* & 0 \end{pmatrix}.$$

Clearly, $(S^*, I^*)$ is locally asymptotically stable, which implying that $E^*$ is locally asymptotically stable.

By Dulac criterion with a Dulac function of the form $1/(SI)$, one can easily show that the two dimensional system does not have nontrivial periodic solutions in the positive quadrant. A careful application of Boincare-Bendixon theorem will allow us to conclude that the endemic steady state is actually globally stable in the positive quadrant.