

Lectures 22, Th., Nov. 2

Reading homework: Chapter 6

1. Applications of Liapunov-LaSalle Theorem in a Food Chain, II. We first recall the definition of Liapunov function. Let

$$x' = f(x), \quad x \in R^n. \quad (1.1)$$

be an n -dimensional system of differential equations. Let $f(x)$ be defined on G^* , an open set in R^n , and let G be a subset of G^* . A function $V(x) : G \rightarrow R$ is said to be a *Liapunov function* for (1.1) on G if

1. V is continuously differentiable at each point $x \in G$, and
2. $\dot{V} = dV/dt|_{(1.1)} = \nabla \cdot V \leq 0$ on G .

The following Liapunov-LaSalle theorem will be the key in our effort in seeking global stability results in some population models.

THEOREM 1.1. (Liapunov-LaSalle) *Let V be a Liapunov function for (1.1) on a region G . Let $E = \{x | \dot{V}(x) = 0, x \in G \cap G^*\}$ and let M be the largest invariant set in E . Then every bounded (for $t \geq 0$) trajectory of (1.1) that remains in G tends to the set M as $t \rightarrow \infty$.*

Consider now the following Lotka-Volterra food chain model

$$x' = x[r(1-x/K) - by] = x(r - ax - by), \quad y' = y(-m + cx - dz), \quad z' = z(ey - f). \quad (1.2)$$

We assume that (1.2) has a positive steady state $E = (x^*, y^*, z^*)$. This is equivalent to say that $re > bf$ and $\frac{c}{a}(r - b\frac{f}{e}) > 0$.

THEOREM 1.2. *In (1.2), if $re > bf$ and $\frac{c}{a}(r - b\frac{f}{e}) > 0$, then all positive solutions tend to $E = (\frac{1}{a}(r - b\frac{f}{e}), \frac{f}{e}, \frac{c}{da}(r - b\frac{f}{e}))$.*

Again, we will proof the above theorem by applying Liapunov-LaSalle theorem. As in the previous example of the Lotka-Volterra predator-prey model, the key step is to construct an appropriate Liapunov function. To this end, we would like to try our luck for a Liapunov function that separate the variables x , y and z . In other word, we assume that

$$V(x, y, z) = V_1(x) + V_2(y) + V_3(z).$$

Let

$$X = x - x^*, \quad Y = y - y^*, \quad Z = z - z^*.$$

Then (1.2) can be rewritten as

$$x' = x(-aX - bY), \quad y' = y(cX - dZ), \quad z' = z(eY). \quad (1.3)$$

The derivative of this function along a solution of (1.2) takes the form of

$$\begin{aligned} \dot{V} &= V_1'(x)x(-aX - bY) + V_2'(y)y(cX - dZ) + V_3'(z)zeY \\ &= -axXV_1'(x) - bV_1'(x)xY + V_2'(y)ycX - V_2'(y)ydZ + V_3'(z)zeY. \end{aligned}$$

We would like to have all the mixed terms in the expression of \dot{V} to cancel each other. In other words, we want

$$bV_1'(x)xY = V_2'(y)ycX, \quad V_2'(y)y dZ = V_3'(z)zeY.$$

This is equivalent to say that

$$f(x) \equiv bV_1'(x)x/X = V_2'(y)yc/Y \equiv g(y)$$

and

$$V_2'(y)y d/Y \equiv \frac{d}{c}g(y) = V_3'(z)ze/Z \equiv h(z).$$

This says a function of x is identical to a different function of another independent variable y and a function of y is identical to a different function of another independent variable z . This can only be true if these functions are constants. For our purpose, we may simply assume that the first constant is 1 ($= f(x) = g(y)$). This yields

$$bV_1'(x) = X/x = 1 - x^*/x, \quad cV_2'(y) = Y/y = 1 - y^*/y.$$

and

$$\frac{ce}{d}V_3'(z) = Z/z = 1 - z^*/z.$$

This in turn suggest that

$$V_1(x) = \frac{1}{b}(x - x^* \ln x), \quad V_2(y) = \frac{1}{c}(y - y^* \ln y), \quad V_3(z) = \frac{d}{ce}(z - z^* \ln z).$$

With these functions, we have

$$\dot{V} = -axXV_1'(x) = -\frac{a}{b}(x - x^*)^2.$$

Hence, $V = \frac{1}{b}(x - x^* \ln x) + \frac{1}{c}(y - y^* \ln y) + \frac{d}{ce}(z - z^* \ln z)$ is indeed a Liapunov function for (1.2). We have $E = \{(x, y, z) : x = x^*, y, z \geq 0\}$. We shall show that the largest invariant set M in E for (1.2) is $\{E\}$. To this end, we assume that $(x(0), y(0), z(0)) \in M$. Since $(x(t), y(t), z(t)) \in E$, we have $x(t) \equiv x^*$ which implies that

$$x'(t) \equiv 0.$$

This in turn yields

$$0 = x'(t) = x^*(r - ax^* - by(t))$$

and hence $y(t) = y^*$. In particular, we must have $y(0) = y^*$. Similarly, we must have $z(0) = z^*$, proving that $M = \{E\}$. This shows that all positive solutions of (1.2) tend to the unique positive steady state E .