# Lectures 21, Tu., Oct. 31

**Reading homework: Chapter 6**

**1. Applications of Liapunov-LaSalle Theorem in Population Dynamics.** We first recall the definition of Liapunov function. Let

$$x' = f(x), \;\; x \in R^n. \tag{1.1}$$

be an n-dimensional system of differential equations. Let $f(x)$ be defined on $G^*$, an open set in $R^n$, and let $G$ be a subset of $G^*$. A function $V(x) : G \to R$ is said to be a *Liapunov function* for (1.1) on $G$ if
1. $V$ is continuously differentiable at each point $x \in G$, and
2. $\dot{V} = dV/dt|_{(1.1)} = \nabla \cdot V \leq 0$ on $G$.

The following Liapunov-LaSalle theorem will be the key in our effort in seeking global stability results in some population models.

Theorem 1.1. *(Liapunov-LaSalle) Let $V$ be a Liapunov function for (1.1) on a region $G$. Let $E = \{x|\dot{V}(x) = 0, x \in G \bigcap G^*\}$ and let $M$ be the largest invariant set in $E$. Then every bounded (for $t \geq 0$) trajectory of (1.1) that remains in $G$ tends to the set $M$ as $t \to \infty$.*

Consider first the following Lotak-Volterra predator-prey model

$$x' = x[r(1 - x/K) - by] = x(r - ax - by), \;\; y' = y(-d + cx). \tag{1.2}$$

By an application of the Dular criterion (with the Dulac function $1/(xy)$) and the Poincare-Bendixson theorem, one can show that

Theorem 1.2. *In (1.2), if $d/c \geq K(= r/a)$, then all positive solutions tend to $(K, 0)$. If $d/c < K(= r/a)$, then all positive solutions tend to the unique positive steady state $E = (d/c, (rc - ad)/(bc))$.*

In this lecture, we will proof the above theorem by applying Liapunov-LaSalle theorem. The key step, naturally, is to construct an appropriate Liapunov function. To this end, we would like to try our luck for a Liapunov function that separate the variable of $x$ from $y$. In other word, we hope that

$$V(x, y) = V_1(x) + V_2(y).$$

We assume first that $d/c < K(= r/a)$ (the other case will be left as a simple homework/exercise). In which case the system has a unique positive steady state $E = (x^*, y^*) = (d/c, (rc - ad)/(bc))$. Let

$$X = x - x^*, \;\; Y = y - y^*.$$

Then (1.2) can be rewritten as

$$x' = x(-aX - bY), \;\; y' = y(cX). \tag{1.3}$$

The derivative of this function along a solution of (1.2) takes the form of

$$\dot{V} = V_1'(x)x(-aX - bY) + V_2'(y)ycX = -axXV_1'(x) - bV_1'(x)xY + V_2'(y)ycX.$$

Remember that we have some flexibility in selecting the functions $V_1(x)$ and $V_2(y)$. A ideal scenario will be to have $V_1(x)$ and $V_2(y)$ such that the mixed terms in the expression of $\dot{V}$ to cancel each other. In other words, we want

$$bV_1'(x)xY = V_2'(y)ycX.$$

This is equivalent to say that

$$f(x) \equiv bV_1'(x)x/X = V_2'(y)yc/Y \equiv g(y).$$

This says a function of $x$ is identical to a different function of another independent variable $y$. This can only be true if these functions are the same constant. For our purpose, we may simply assume that this constant is 1. This yields

$$bV_1'(x) = X/x = 1 - x^*/x, \;\; cV_2'(y) = Y/y = 1 - y^*/y.$$

This is turn suggest that

$$V_1(x) = \frac{1}{b}(x - x^* \ln x), \;\; V_2(y) = \frac{1}{c}(y - y^* \ln y).$$

With this pair of functions, we have

$$\dot{V} = -axXV_1'(x) = -\frac{a}{b}(x - x^*)^2.$$

Hence, $V = \frac{1}{b}(x - x^* \ln x) + \frac{1}{c}(y - y^* \ln y)$ is indeed a Liapunov function for (1.2). We have $E = \{(x, y) : x = x^*, y \geq 0\}$. We shall show that the largest invariant set $M$ in $E$ for (1.2) is $\{E\}$. To this end, we assume that $(x(0), y(0)) \in M$. Since $(x(t), y(t)) \in E$, we have $x(t) \equiv x^*$ which implies that

$$x'(t) \equiv 0.$$

This in turn yields

$$0 = x'(t) = x^*(r - ax^* - by(t))$$

and hence $y(t) = y^*$. In particular, we must have $y(0) = y^*$, proving that $M = \{E\}$. This shows that all positive solutions of (1.2) tend to the unique positive steady state $E = (d/c, (rc - ad)/(bc))$ when $d/c < K(= r/a)$.