

## Lectures 20, Th., Oct. 26

Reading homework: Section 2.6 and 2.7 of the book of Waltman (A Second Course in Elementary Differential Equations, Dover, 2004, \$11.82 at walmart.com)

**1. Elementary Liapunov Stability Theory.** (This lecture is adapted from Section 2.6 of Waltman, Second Course in Elementary Differential Equations, Dover, 2004)

The most often quoted Liapunov stability theorem in elementary differential equation textbooks and engineering applications take the form of

THEOREM 1.1. Let  $V(x, y)$  be a differentiable function that satisfies

- i).  $V(x, y) > 0, (x, y) \neq (0, 0)$ ,
- ii).  $V(0, 0) = 0$
- iii).  $\dot{V} = dV/dt \leq -kV$

for some constant  $k > 0$ . Then every solution of

$$x' = f(x, y), \quad y' = g(x, y)$$

satisfies

$$\lim_{t \rightarrow \infty} (x(t), y(t)) = (0, 0).$$

This is a theorem with very limited application in mathematical biology, since the hypotheses on iii) on  $V(x, y)$  is very strong and inconvenient. Here is a slightly nontrivial application of this theorem. Consider

$$x' = y - x, \quad y' = -y - x^3.$$

and the function  $V(x, y) = (1/2)y^2 + (1/4)x^4$ . Then

$$dV/dt = x^3y - x^4 - x^3y - y^2 = -x^4 - y^2 \leq -2V.$$

The above theorem applies, and all solutions tend to zero as  $t$  tends to infinity.

The more modern, popular and general Liapunov-LaSalle theorem presented below, provides a setting appropriate for the applications in studying the global dynamics of many interesting population models.

Let

$$x' = f(x), \quad x \in R^n. \tag{1.1}$$

be an  $n$ -dimensional system of differential equations. Let  $f(x)$  be defined on  $G^*$ , an open set in  $R^n$ , and let  $G$  be a subset of  $G^*$ . A function  $V(x) : G \rightarrow R$  is said to be a *Liapunov function* for (1.1) on  $G$  if

1.  $V$  is continuously differentiable at each point  $x \in G$ , and
2.  $\dot{V} = dV/dt|_{(1.1)} = \nabla \cdot V \leq 0$  on  $G$ .

The power of the Liapunov method is that the derivative of  $V$  with respect to  $t$  can be computed along but without a knowledge of solutions of the differential equation.

In order to state the celebrated Liapunov-LaSalle theorem, it is necessary to introduce several definitions. First, a set  $\Omega$  is said to be positively invariant with respect to (1.1) if the trajectory through  $x_0 \in G$  at  $t = 0$  remains in  $\Omega$  for all  $t \geq 0$ . Negatively invariant is defined by replacing  $t \geq 0$  by  $t \leq 0$ . A set is invariant if it is positively and negatively invariant. Critical points, for example, are positively and negatively invariant; the set of points on a periodic orbit is also invariant. The distance between a point  $p$  and a closed set  $A$ , both in  $R^n$ , is defined by

$$d(p, A) = \min_x \{|x - p| : x \in A\}.$$

Here  $|\cdot|$  is a norm. Notice that a trajectory may approach a set of points without approaching any specific point of the set as illustrated by the trajectories that spiral toward a periodic trajectory (limit cycle).

The following Liapunov-LaSalle theorem is a special case of what is commonly called the LaSalle corollary to the Liapunov stability theorem. (Note that  $\overline{G}$  means the closure of  $G$ .)

**THEOREM 1.2. (Liapunov-LaSalle)** *Let  $V$  be a Liapunov function for (1.1) on a region  $G$ . Let  $E = \{x | \dot{V}(x) = 0, x \in G \cap G^*\}$  and let  $M$  be the largest invariant set in  $E$ . Then every bounded (for  $t \geq 0$ ) trajectory of (1.1) that remains in  $G$  tends to the set  $M$  as  $t \rightarrow \infty$ .*

The proof of this theorem can be found in the popular reference book by Hale (page 316, Theorem 1.3 in J. K. Hale, Ordinary Differential Equations, Krieger Publishing Co., 1980). We illustrate it with several examples.

Consider

$$x' = -xy^2, \quad y' = -x^4y. \tag{1.2}$$

and let  $V = x^2 + y^2$  and  $G = G^* = R^2$ . Then,

$$\dot{V} = 2xx' + 2yy' = -2x^2y^2(1 + x^2) \leq 0.$$

$E = \{(x, y) : x = 0 \text{ or } y = 0\}$ , and all of  $E$ 's invariant. Therefore,  $M = E$  and every bounded trajectory approaches a subset of the  $x$ - or the  $y$ -axis (or possibly both).

Consider the system

$$x' = -y, \quad y' = -y + g(x). \tag{1.3}$$

Here  $xg(x) > 0, x \neq 0$ . Let

$$V(x, y) = \frac{y^2}{2} + \int_0^x g(s)ds$$

and let  $G = G^* = R^2$ . Then

$$\dot{V} = yy' + g(x)x' = -y^2 + yg(x) + g(x)(-y) = -y^2 \leq 0.$$

Therefore,  $E = \{(x, y) : y = 0\}$ . Let  $(s, 0) \in M$ , the largest invariant set in  $E$ . Then, for a trajectory through  $(s, 0)$  to remain in  $M$ , it must satisfy the system

$$x' = 0, \quad y' = g(x) = 0,$$

since to remain in  $E$  it must be the case that  $y(t) = 0$  and hence  $y'(t) = 0$ . By the first equation,  $x'(t) = 0$  or  $x(t) = C$ , a constant. Therefore,  $g(C) = 0$ , but the only

value of  $x$  where  $g(x) = 0$  is  $x = 0$ . Thus, the only invariant set in  $E$  is the point  $(0, 0)$  and by the Liapunov-LaSalle theorem, all bounded solutions tend to the origin.

Sometimes the Liapunov-LaSalle theorem can be used to determine the region of attraction for a critical point. Consider the system

$$x' = -y, \quad y' = x + y^3 - y. \quad (1.4)$$

The only critical point is the origin. Let  $V(x, y) = x^2 + y^2$  and compute

$$\dot{V} = -2xy + 2xy + 2y^4 - 2y^2 = 2y^2(y^2 - 1).$$

Therefore,  $V(x, y)$  will be a Liapunov function for (1.4) if  $|y| < 1$ . Consider the interior of the level curve  $x^2 + y^2 = 1$  (the largest level curve in the region  $|y| \leq 1$ ) as the region  $G$ . Along any solution that starts in this region,  $\dot{V}(x, y) < 0$  if  $y \neq 0$ . The set  $E = \{(x, y) : y = 0 \text{ or } y = \pm 1\}$  and  $M = \{(0, 0)\}$ ; so all solutions that start in  $G$  tend to the origin. Thus,  $G$  represents a set of points “attracted” to the origin.

In the next lecture, we will introduce a simple yet effective way of constructing custom-made Liapunov functions for some popular population models.