Lectures 16, Th., Oct. 12

Reading homework: chapter 5

1. Global stability results for the chemostat model. We reviewed section 5.10 and proved two global stability results for the chemostat model

$$\begin{cases} N' &= \alpha_1 \dfrac{CN}{1+C} - N, \\ C' &= -\dfrac{CN}{1+C} - C + \alpha_2. \end{cases} \tag{1.1}$$

THEOREM 1. *(Global stability of $E_2 = (\bar{N}_2, \bar{C}_2)$) If $C(0) < \alpha_2$, $\alpha_1 > 1$ and $\alpha_2 < \frac{1}{\alpha_1 - 1}$, then $\lim_{t\to\infty}(N, C) = (\bar{N}_2, \bar{C}_2)$.*

Proof. We show first that $C(t) < \alpha_2$ for $t > 0$. If not, then there is a $t_1 > 0$, such that $C(t_1) = \alpha_2$ and $C'(t_1) \geq 0$. However from the C equation, we see that $C'(t_1) = -\dfrac{C(t_1)N(t_1)}{1 + C(t_1)} < 0$, a contradiction. Observe that $\alpha_2 < \frac{1}{\alpha_1 - 1}$ implies that $\alpha_2\alpha_1 < 1 + \alpha_2$. Let $\theta = \alpha_2\alpha_1/(1 + \alpha_2)$. Then we have

$$N' = \alpha_1 \frac{CN}{1+C} - N < -(1-\theta)N.$$

This shows that $\lim_{t\to\infty} N = 0$. In particular, for any $\varepsilon > 0$, there is a $T > 0$ such that $N(t) < \varepsilon$ for $t > T$. Hence for $t > T$, $-C + \alpha_2 - \varepsilon < C' < -C + \alpha_2$. Standard comparison arguments (compare to $C' = -C+\alpha_2$ and $C' = -C+\alpha_2-\varepsilon$) yield that for large $t > T$, $C(t) \in [\alpha_2 - \varepsilon, \alpha_2]$. Letting $\varepsilon \to 0$ yields that $\lim_{t\to\infty}(N, C) = (\bar{N}_2, \bar{C}_2)$. ∎

THEOREM 2. *(Global stability of $E_1 = (\bar{N}_1, \bar{C}_1)$) If $C(0) < \alpha_2$, $\alpha_1 > 1$ and $\alpha_2 > \frac{1}{\alpha_1 - 1}$, then $\lim_{t\to\infty}(N, C) = (\bar{N}_1, \bar{C}_1)$.*

Proof. Let $Z(t) = N(t) + \alpha_1 C(t)$. Then

$$Z'(t) = \alpha_1\alpha_2 - Z(t).$$

Hence $Z(t) = \alpha_1\alpha_2 + (Z(0) - \alpha_1\alpha_2)e^{-t}$. This shows that $Z(t)$ tends to $\alpha_1\alpha_2$ monotonically. Hence the global dynamics is the same as that of (by a result of H. R. THIEME, *Mathematics in Population Biology*, Princeton Series in Theoretical and Computational Biology, Princeton University Press (2003)) the limiting system of $N(t) = \alpha_1(\alpha_2 - C(t))$ and $N' = -\alpha_1 C' = \alpha_1(\dfrac{\alpha_1 C}{1+C} - 1)\alpha_1(\alpha_2 - C(t)) = \alpha_1 \dfrac{(\alpha_1 - 1)(C - \bar{C}_1)}{1 + C}(\alpha_2 - C(t))$. It is easy to see that $\lim_{t\to\infty} C = \bar{C}_1$. This implies that $\lim_{t\to\infty} N = \bar{N}_1$. ∎