Lectures 14, Th., Oct. 5

Reading homework: chapter 4

1. A chemostat model. We covered sections 4.6-4.7. In addition, we have shown that the solutions of the model (19a)-(19b) with positive initial values are positive and bounded. Standard existence and uniqueness theorems of MAT 475 (or MAT 574) ensure that such solution exist and unique for $t > 0$.

THEOREM 1. *(Positivity) The solutions of the model (19a)-(19b) with positive initial values are positive for $t > 0$.*

Proof. If not, then there is a $t_1 > 0$, such that $N(t_1)C(t_1) = 0$ and $N(t)C(t) > 0$ for $t \in [0, t_1)$. Assume first that $C(t_1) = 0$. Then $C'(t_1) \leq 0$. However, (19b) implies that $C'(t_1) = \alpha_2 > 0$, a contradiction. In the rest of this proof, we show that it is impossible that $N(t_1) = 0$. From (19a) and the fact that it is impossible that $C(t_1) = 0$, we see that for $t \in [0, t_1)$, we have that

$$N'(t) \geq -N(t)$$

which yields that for $t \in [0, t_1)$,

$$N(t) \geq N(0)e^{-t} > N(0)e^{-t_1}.$$

Since $N(t)$ is continuous on $[0, t_1]$, we see that $N(t_1) \geq N(0)e^{-t_1} > 0$. This is a contradiction. ∎

THEOREM 2. *(Boundedness) The solutions of the model (19a)-(19b) with positive initial values are bounded for $t > 0$.*

Proof. Let $Z(t) = N(t) + \alpha_1 C(t)$. Then

$$Z'(t) = \alpha_1\alpha_2 - Z(t).$$

Hence $Z(t) = \alpha_1\alpha_2 + (Z(0) - \alpha_1\alpha_2)e^{-t} \leq \max\{\alpha_1\alpha_2, Z(0)\}$. This shows that both $N(t)$ and $C(t)$ are bounded for $t > 0$. ∎