10900. *Proposed by Gordon Rice, Davis, CA.* It is clear from the law of cosines that every angle that occurs in a triangle with integer sides has a rational cosine. Is the converse true? Does every angle between 0 and π with a rational cosine occur in some triangle with integer sides?

Solution: Yes. Let the sides of a triangle be a, b, and c, and suppose $\cos\theta = r$ is rational, with θ between 0 and π. If $r = 0$, we can take $a = 3$, $b = 4$, and $c = 5$. If $r < 0$, then let

$$\left\{ \begin{array}{rcl} a &=& 2A \\ b &=& B \\ c &=& B \end{array} \right\}, \quad \text{where} \quad r = \frac{A}{B} \quad \text{and} \quad A, B > 0.$$

If $r < 0$, then let

$$\left\{ \begin{array}{rcl} a &=& B^2 - (A+1)^2 \\ b &=& 2B \\ c &=& B^2 - A^2 + 1 \end{array} \right\}, \quad \text{where} \quad r = \frac{-A}{B}, \quad A, B > 0, \quad \text{and} \quad B > A + 1.$$

The condition that $B > A + 1$ can be met in the last case by noting that r need not be expressed in lowest terms; so if $r = -\frac{1}{2}$, then we can let $A = 2$ and $B = 4$.

The validity of the result follows from a straightfoward calculation. In each case, c is the length of the side opposite the angle θ.

A brief note how these values were calculated follows. The case where $r > 0$ is based on the right triangle with sides A, B, and $\sqrt{B^2 - A^2}$. The vertex adjacent to the angle θ was reflected about the opposite side, to make sure that the triangle has integral sides.

The case where $r < 0$ was trickier. Integral values of a, b, and c needed to be found so that

$$-\frac{A}{B} = \frac{a^2 + b^2 - c^2}{2ab}.$$

The initial guess $b = B$ impiles that

$$-2aA = a^2 + B^2 - c^2.$$

Rearrangement and completing the square yields

$$A^2 - B^2 = (A + a)^2 - c^2.$$

If the left-hand side is odd, then $A + a = c - 1$ is reasonable, because then

$$A^2 - B^2 = -2c + 1, \text{ or } \quad c = \frac{B^2 - A^2 + 1}{2},$$

making c integral. Consequently,

$$a = c - 1 - A = \frac{B^2 - A^2 - 1}{2} - A$$

is then also an integer. These formulas were then doubled to make sure that a, b, and c are integers for any choice of A and B with $B > A > 0$ (i.e., $-1 < r < 0$). Not every such A and B will work, because $a \leq 0$ if $B \leq A + 1$; hence the condition that $B > A + 1$ must be imposed.

This finishes the problem, but one observation springs to mind: If $\cos\theta$ is rational and positive, then θ appears as an angle in an *isosceles* triangle with integral sides. Is this true for any θ with $\cos\theta$ rational?

Unfortunately, the answer to this question is no. In fact, if $\theta = \frac{\pi}{2}$ (a right angle), then in order for the isosceles version to hold, we must have $a = b$ and $c = a\sqrt{2}$. Since the square root of two is irrational, no such isosceles triangle can be found. In fact, whenever $\sqrt{2 - 2\cos\theta}$ is irrational, no such triangle will exist (again, because if $\theta > \frac{\pi}{2}$, then $a = b$).

However, this square root can be approximated by a rational number r to within any real $\varepsilon > 0$, and the angle θ' will approximate θ, by the continuity of the cosine function. This means that the following result holds (even if $\cos\theta$ is irrational):

Proposition. *Let θ be an angle between 0 and π, and let $\varepsilon > 0$ be a real number. Then there exists an isosceles triangle with integral sides, such that one of its angles approximates θ to within ε. Hence, the set of angles in isosceles integral triangles is dense in $[0, \pi]$.*