

11333. Proposed by Pablo Fernández Refolio, Universidad Autónoma de Madrid, Madrid, Spain. Show that

$$\prod_{n=2}^{\infty} \left(\left(\frac{n^2-1}{n^2} \right)^{2(n^2-1)} \left(\frac{n+1}{n-1} \right)^n \right) = \pi.$$

Solution by Christopher Carl Heckman, Arizona State University, Tempe, AZ: Let

$$a_n = \left(\frac{n^2-1}{n^2} \right)^{2(n^2-1)} \left(\frac{n+1}{n-1} \right)^n = \frac{(n-1)^{2n^2-n-2} (n+1)^{2n^2+n-2}}{n^{4n^2-4}}.$$

Calculating $\prod_{n=2}^N a_n$ for small values of N suggests that

$$\prod_{n=2}^N a_n = \frac{1}{2} [(N-1)!]^2 \cdot \frac{(N+1)^{2N^2+N-2}}{N^{2N^2+3N-3}}$$

when $N \geq 2$, which can be proven by mathematical induction on N .

Stirling's approximation for $n!$ can be used when taking the limit as $N \rightarrow +\infty$, since the factorial is being raised to a constant power. We then obtain

$$\begin{aligned} \prod_{n=2}^{\infty} a_n &= \lim_{n \rightarrow +\infty} \frac{1}{2} [(n-1)!]^2 \cdot \frac{(n+1)^{2n^2+n-2}}{n^{2n^2+3n-3}} = \lim_{n \rightarrow +\infty} \frac{1}{2} \cdot \left[\left(\frac{n-1}{e} \right)^{n-1} \cdot \sqrt{2\pi(n-1)} \right]^2 \cdot \frac{(n+1)^{2n^2+n-2}}{n^{2n^2+3n-3}} \\ &= \lim_{n \rightarrow +\infty} \frac{1}{2} \cdot \frac{(n-1)^{2n-1}}{e^{2n-2}} \cdot 2\pi \cdot \left(\frac{n+1}{n} \right)^{2n^2} \cdot \left(\frac{n+1}{n} \right)^n \cdot \frac{1}{n^{2n-1}} \cdot \left(\frac{n}{n+1} \right)^2 \\ &= \lim_{n \rightarrow +\infty} \pi \cdot \left(1 - \frac{1}{n} \right)^{2n-1} \cdot \left(1 + \frac{1}{n} \right)^n \cdot \left(\frac{n}{n+1} \right)^2 \cdot e^2 \cdot e^{-2n} \left(1 + \frac{1}{n} \right)^{2n^2} \\ &= \pi \cdot \lim_{n \rightarrow +\infty} \left(1 - \frac{1}{n} \right)^{2n-1} \cdot \lim_{n \rightarrow +\infty} \left(1 + \frac{1}{n} \right)^n \cdot \lim_{n \rightarrow +\infty} \left(\frac{n}{n+1} \right)^2 \cdot e^2 \cdot \lim_{n \rightarrow +\infty} e^{-2n} \left(1 + \frac{1}{n} \right)^{2n^2} \\ &= \pi \cdot e^{-2} \cdot e \cdot 1 \cdot e^2 \cdot e^{-1} = \pi, \end{aligned}$$

since each of the simpler limits exists. Most of the limits above are comparable in difficulty to exercises from a Calculus class. The most challenging limit is $L = \lim_{x \rightarrow +\infty} e^{-2x} \left(1 + \frac{1}{x} \right)^{2x^2}$, which is handled with the following finesse (and L'Hospital's Rule):

$$\begin{aligned} \ln L &= \lim_{x \rightarrow +\infty} \ln e^{-2x} \left(1 + \frac{1}{x} \right)^{2x^2} = \lim_{x \rightarrow +\infty} \left[-2x + 2x^2 \ln \left(1 + \frac{1}{x} \right) \right] = \lim_{x \rightarrow +\infty} \frac{-\frac{2}{x} + 2 \ln \left(1 + \frac{1}{x} \right)}{\frac{1}{x^2}} \\ &= \lim_{x \rightarrow +\infty} \frac{\frac{2}{x^2} + \frac{2}{1+1/x} \cdot -\frac{1}{x^2}}{\frac{-2}{x^3}} = \lim_{x \rightarrow +\infty} \left(-x + \frac{x}{1+1/x} \right) = \lim_{x \rightarrow +\infty} \frac{-x}{x+1} = -1, \end{aligned}$$

so $L = e^{-1}$, and the result holds.

Title this solution: "Using Factorials to Bake a Pi."