11241. *Proposed by Roberto Tauraso, Università di Roma "Tor Vergata", Rome, Italy.* Find a closed formula for

$$\sum_{k=0}^{n} 2^{n-k} \sum_{x\in S[k,n]} \prod_{i=1}^{k+1} F_{1+2x_i},$$

where F_n denotes the nth Fibonacci number (that is, $F_0 = 0$, $F_1 = 1$, and $F_j = F_{j-1} + F_{j-2}$ when $j \geq 2$) and $S[k,n]$ is the set of all $(k+1)$-tuples of nonnegative integers that sum to $n-k$.

Solution by Christopher Carl Heckman, Arizona State Univeriesty, Tempe, AZ: We proceed using generating functions; that is, let

$$a_n = \sum_{k=0}^{n} 2^{-k} \sum_{x\in S[k,n]} \prod_{i=1}^{k+1} F_{1+2x_i},$$

and

$$A(z) = a_0 + a_1 z + a_2 z^2 + a_3 z^3 + \cdots$$

as a formal power series.

If we fix k for the moment, then the coefficient of z^m in $(F_1 + F_3 z + F_5 z^2 + F_7 z^3 + \cdots)^{k+1}$ is $\displaystyle\sum_{x\in S[k,m+k]} \prod_{i=1}^{k+1} F_{1+2x_i}$: if the expression above is multiplied out, it will consist of the sum of a bunch of terms of the form

$$(F_{2y_1+1} z^{y_1})(F_{2y_2+1} z^{y_2}) \cdots (F_{2y_{k+1}+1} z^{y_{k+1}}),$$

where y_i is the power of z taken from the ith factor. In order to end up with an exponent of z^m, it is necessary and sufficient to have $y_1 + y_2 + \cdots + y_{k+1} = m$. Since $y_i \geq 0$ as well, this means y is one of the elements of $S[k, k+m]$. When we include other terms, we get the rest of the elements of $S[k, k+m]$.

Next, the coefficient of z^m needs to be transferred to a coefficient of z^{m+k}, which can be done by multiplying by z^k. Simultaneously, we multiply this coefficient by 2^{-k}. What we have shown is that the generating function of

$$2^{-k} \sum_{x\in S[k,n]} \prod_{i=1}^{k+1} F_{1+2x_i}$$

is $2^{-k} z^k (F_1 + F_3 z + F_5 z^2 + F_7 z^3 + \cdots)^{k+1}$. Now, we sum over all k to get the generating function for a_n:

$$A(z) = \sum_{k=0}^{\infty} 2^{-k} z^k (F_1 + F_3 z + F_5 z^2 + F_7 z^3 + \cdots)^{k+1}.$$

Next, we look for a closed form for $A(z)$. It is well-known that the generating function for all of the Fibonacci numbers is

$$\Phi_{\text{all}}(z) = F_0 + F_1 z + F_2 z^2 + \cdots = \frac{z}{1-z-z^2},$$

and since $F_0 = 0$, we can divide both sides of this equation by z to get

$$F_1 + F_2 z + F_3 z^2 + \cdots = \frac{1}{1-z-z^2},$$

and

$$\begin{aligned} F_1 + F_3 z^2 + F_5 z^4 + \cdots &= \frac{1}{2}\left[(F_1 + F_2 z + F_3 z^2 + F_4 z^3 + \cdots) + (F_1 - F_2 z + F_3 z^2 - F_4 z^3 + \cdots)\right] \\ &= \frac{1}{2}\left(\frac{1}{1-z-z^2} + \frac{1}{1+z-z^2}\right) = \frac{1-z^2}{1-3z^2+z^4}. \end{aligned}$$

Replacing z with $\sqrt{z}$ yields

$$\Phi_{\text{odd}}(z) \equiv F_1 + F_3 z + F_5 z^2 + \cdots = \frac{1-z}{1-3z+z^2}.$$

Now $A(z)$ is a geometric series, so $A(z)$ is

$$\sum_{k=0}^{\infty} 2^{-k} z^k (\Phi(z))^{k+1} = \frac{\Phi(z)}{1-\frac{z}{2}\cdot\Phi(z)} = \frac{2-2z}{2-7z+3z^2} = \frac{2/5}{2-z} + \frac{4/5}{1-3z} = \frac{1/5}{1-z/2} + \frac{4/5}{1-3z}.$$

Since the generating function of r^n is $\dfrac{1}{1-rz}$, this implies that $a_n = \dfrac{1}{5}\left(\dfrac{1}{2}\right)^n + \dfrac{4}{5}\cdot 3^n$, and the original expression equals 2^n times this, or $\dfrac{1}{5}(1+4\cdot 6^n)$.

A natural generalization of this problem is that of replacing 2 by some real number α. The hardest part of the proof is finding the partial fraction decomposition of $A(z)$ above; it is nice when the denominator factors. If 2 is replaced with α, and the procedure above is followed, the denominator of $A(z)$ turns out to be

$$(\alpha+1)z^2 + (-3\alpha-1)z + \alpha.$$

(The special case where $\alpha = -1$ has to be dealt with separately, of course.) This quadratic factors iff its descriminant is a perfect square; that is we need

$$(-3\alpha-1)^2 - 4\alpha(\alpha+1) = M^2.$$

If we solve this quadratic for α in terms of M, we get $\alpha = \dfrac{-1\pm\sqrt{5M^2-4}}{5}$. In the "nice" cases, α is a rational number and $5M^2-4$ is a perfect square.

The nonnegative integral solutions to the equation $5M^2 - 4 = N^2$ are well known; it turns out that $M = F_m$ (the mth Fibonacci number) and $N = L_m$ (the mth Lucas number*), for some nonnegative odd integer m.

The Fibonacci numbers and Lucas numbers get involved with the solution at this point. The derivation is straightforward but messy. The result (checked with Maple) is as follows:

Proposition 1. *Let* $F_{\alpha,\text{odd}}(n) = \sum_{k=0}^{n} \alpha^{n-k} \sum_{x\in S[k,n]} \prod_{i=1}^{k+1} F_{1+2x_i}$. *Then, for all* $n > 0$,

(a) $F_{-1,\text{odd}}(n) = -\frac{1}{2}(-2)^n$;

(b) *if* $\alpha = \dfrac{-1+L_m}{5}$, *where* m *is an odd integer, then*

$$F_{\alpha,\text{odd}}(n) = \frac{(6-L_m-5F_m)(-2-3L_m+5F_m)}{20F_m(L_m+4)}\cdot\left(\frac{2(L_m-1)(L_m+4)}{5(2+3L_m+5F_m)}\right)^n + \frac{(6-L_m+5F_m)(2+3L_m+5F_m)}{20F_m(L_m+4)}\cdot\left(\frac{2(L_m-1)(L_m+4)}{5(2+3L_m-5F_m)}\right)^n; \quad \textit{and}$$

(c) *if* $\alpha = \dfrac{-1-L_m}{5}$, *where* m *is an odd integer, and* $\alpha \neq -1$,

$$F_{\alpha,\text{odd}}(n) = \frac{(6+L_m+5F_m)(-2+3L_m-5F_m)}{20F_m(L_m-4)}\cdot\left(\frac{2(L_m+1)(L_m-4)}{5(2-3L_m-5F_m)}\right)^n + \frac{(-6-L_m+5F_m)(-2+3L_m+5F_m)}{20F_m(L_m-4)}\cdot\left(\frac{2(L_m+1)(L_m-4)}{5(2-3L_m+5F_m)}\right)^n.$$

Note that the original problem is a special case of Proposition 1(b), where $m = 5$.

What if the even terms of the Fibonacci numbers are used in the original problem? Then the following hold (provided there are no typos; the mathematics was verified using Maple).

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* The Lucas numbers satisfy the Fibonacci relation $L_n = L_{n-1} + L_{n-2}$, but start off differently: $L_1 = 1$ and $L_2 = 3$.

Proposition 2. *Let* $F_{\alpha,\text{even}}(n) = \sum_{k=0}^{n} \alpha^{n-k} \sum_{x \in S[k,n]} \prod_{i=1}^{k+1} F_{2x_i}$*; then, for all* $n > 0$*,*

(a) $F_{1,\text{even}}(n) = 3^{n-1}$ *and* $F_{-1,\text{even}}(n) = (-2)^n - (-1)^n$*;*

(b) *if* $\alpha = \dfrac{-2 + L_m}{5}$*, where* m *is an even integer, then*

$$F_{\alpha,\text{even}}(n) = \frac{(L_m + 8 + 5F_m)(8 + L_m - 5F_m)(L_m - 2)}{20F_m(L_m + 3)(L_m - 7)} \cdot \left(\frac{2(L_m - 2)(L_m - 7)}{5(-6 + 3L_m + 5F_m)}\right)^n - \frac{(8 + L_m + 5F_m)(8 + L_m - 5F_m)(L_m - 2)}{20F_m(L_m + 3)(L_m - 7)} \cdot \left(\frac{2(L_m - 2)(L_m - 7)}{5(-6 + 3L_m - 5F_m)}\right)^n; \quad \textit{and}$$

(c) *if* $\alpha = \dfrac{-2 - L_m}{5}$*, and* $\alpha \neq -1$*, where* m *is an even integer, then*

$$F_{\alpha,\text{even}}(n) = \frac{(8 - L_m - 5F_m)(8 - L_m + 5F_m)(L_m + 2)}{20F_m(L_m - 3)(L_m + 7)} \cdot \left(\frac{2(L_m + 2)(L_m + 7)}{5(-6 - 3L_m - 5F_m)}\right)^n - \frac{(8 - L_m - 5F_m)(8 - L_m + 5F_m)(L_m + 2)}{20F_m(L_m - 3)(L_m + 7)} \cdot \left(\frac{2(L_m + 2)(L_m + 7)}{5(-6 - 3L_m + 5F_m)}\right)^n.$$

If the full Fibonacci sequence is used (that is, F_{x_i} is substituted for F_{2x_1+1})

Proposition 3. *Let* $F_{\alpha,\text{all}}(n) = \sum_{k=0}^{n} \alpha^{n-k} \sum_{x \in S[k,n]} \prod_{i=1}^{k+1} F_{x_i}$*; then, for all* $n > 0$*,*

(a) $F_{-1,\text{all}}(n) = (-1)^n$*;*

(b) *if* $\alpha = \dfrac{-2 + L_m}{5}$*, where* m *is an even integer, then*

$$F_{\alpha,\text{all}}(n) = \frac{(L_m - 2)(4 + 3L_m + 5F_m)(4 + 3L_m - 5F_m)}{20F_m(L_m + 3)^2} \cdot \left(\frac{2(L_m - 2)(L_m + 3)}{5(2 - L_m + 5F_m)}\right)^n - \frac{(L_m - 2)(4 + 3L_m + 5F_m)(4 + 3L_m - 5F_m)}{20F_m(L_m + 3)^2} \cdot \left(\frac{2(L_m - 2)(L_m + 3)}{5(2 - L_m - 5F_m)}\right)^n; \quad \textit{and}$$

(c) *if* $\alpha = \dfrac{-2 - L_m}{5}$*, where* m *is an even integer, and* $\alpha \neq -1$*, then*

$$F_{\alpha,\text{all}}(n) = \frac{(L_m + 2)(4 - 3L_m - 5F_m)(4 - 3L_m + 5F_m)}{20F_m(L_m - 3)^2} \cdot \left(\frac{2(L_m + 2)(L_m - 3)}{5(2 + L_m - 5F_m)}\right)^n - \frac{(L_m + 2)(4 - 3L_m - 5F_m)(4 - 3L_m - 5F_m)}{20F_m(L_m - 3)^2} \cdot \left(\frac{2(L_m + 2)(L_m - 3)}{5(2 + L_m + 5F_m)}\right)^n.$$