11212. *Proposed by David Beckwith, Sag Harbor, NY.* Show that for an arbitrary positive integer n

$$\sum_{r=0}^{n}(-1)^r\binom{n}{r}\binom{2n-2r}{n-1}=0.$$

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(1) The first proof combines the Pigeonhole Principle and the Inclusion-Exclusion formula.

Let

$$\begin{aligned} S &= \{1,2,\ldots,n\}\times\{1,2\}, \\ \mathcal{T}_i &= \{T\subseteq S : |T|=n+1 \text{ and } (i,1),(i,2)\in T\}, \quad \text{and} \\ \mathcal{T} &= \bigcup_{i=1}^{n}\mathcal{T}_i. \end{aligned}$$

The Inclusion-Exclusion formula implies that

$$\begin{aligned} |\mathcal{T}| = |\mathcal{T}_1\cup\mathcal{T}_2\cup\cdots\cup\mathcal{T}_n| &= \sum_{1\le i\le n}|\mathcal{T}_i| - \sum_{1\le i<j\le n}|\mathcal{T}_i\cap\mathcal{T}_j| + \cdots + (-1)^{n-1}|\mathcal{T}_1\cap\mathcal{T}_2\cap\cdots\cap\mathcal{T}_n| \\ &= \binom{n}{1}|\mathcal{T}_1| - \binom{n}{2}|\mathcal{T}_1\cap\mathcal{T}_2| + \cdots + (-1)^{n-1}\binom{n}{n}|\mathcal{T}_1\cap\mathcal{T}_2\cap\cdots\cap\mathcal{T}_n| \end{aligned}$$

To find $|\mathcal{T}_1\cap\mathcal{T}_2\cap\cdots\cap\mathcal{T}_r|$, note that if $T\in\mathcal{T}_1\cap\mathcal{T}_2\cap\cdots\cap\mathcal{T}_r$, then $(1,1),(1,2),\ldots,(r,1),(r,2)\in T$. The rest of the elements of T have to be chosen from the set $\{r+1,\ldots,n\}\times\{1,2\}$, which has $2n-2r$ elements. There are $\binom{2n-2r}{n+1-2r}$ ways to choose the remaining elements of T; note that this formula works even if $2r>n+1$, since in that case, $\binom{2n-2r}{n+1-2r}=0$. Thus

$$|\mathcal{T}_1\cap\mathcal{T}_2\cap\cdots\cap\mathcal{T}_r| = \binom{2n-2r}{n+1-2r} = \binom{2n-2r}{(2n-2r)-(n+1-2r)} = \binom{2n-2r}{n-1},$$

and

$$|\mathcal{T}| = \binom{n}{1}\binom{2n-2\cdot 1}{n-1} - \binom{n}{2}\binom{2n-2\cdot 2}{n-1} + \cdots + (-1)^{n-1}\binom{n}{n}\binom{2n-2\cdot n}{n-1} = \sum_{r=1}^{n}(-1)^{r-1}\binom{n}{r}\binom{2n-2r}{n-1}.$$

The Pigeonhole Principle states that, if T is a subset of S with $n+1$ elements, there will be at least one integer k such that $(k,1),(k,2)\in T$. This implies that $|\mathcal{T}| = \binom{2n}{n+1} = (-1)^0\binom{n}{0}\binom{2n-2(0)}{n-1}$, that is:

$$(-1)^0\binom{n}{0}\binom{2n-2(0)}{n-1} = \sum_{r=1}^{n}(-1)^{r-1}\binom{n}{r}\binom{2n-2r}{n-1},$$

which can be rearranged into

$$\sum_{r=0}^{n}(-1)^r\binom{n}{r}\binom{2n-2r}{n-1}=0.$$

(2) The second proof uses Sister Celine's method, as described in *A=B*, written by Marko Petkovšek, Herbert Wilf and Doron Zeilberger. In fact, this is problem 3(a) in Section 4.6 (p. 72) with $a=-1$. Let $F(n,r)=(-1)^r\binom{n}{r}\binom{2n-2r}{n-1}$ and $f(n)=\sum_{r=0}^{n}F(n,r)$. Then

$$\frac{F(n-2,r-1)}{F(n,r)} = \frac{(n-2)r}{2n(2r-2n+1)} \quad\text{and}\quad \frac{F(n-1,r)}{F(n,r)} = \frac{(n-r)(n-1)(n+1-2r)}{n(2n-1-2r)(2n-2r)}.$$

It is easy to check that

$$(4n^2 - 4n) \cdot \frac{(n-2)r}{2n(2r-2n+1)} + (2n-4n^2) \cdot \frac{(n-r)(n-1)(n+1-2r)}{n(2n-1-2r)(2n-2r)} + (n^2-1) = 0,$$

which means that

$$(4n^2 - 4n)F(n-2, r-1) + (2n-4n^2)F(n-1, r) + (n^2-1)F(n, r) = 0.$$

Summing up this equation from $r = -3$ to $r = n$, we find out that

$$(4n^2 - 4n)f(n-2) + (2n-4n^2)f(n-1) + (n^2-1)f(n) = 0, \tag{E1}$$

for all $n \geq 1$.

Now the proposition $f(n) = 0$ can be proved by mathematical induction. It is easily checked that $f(1) = 0$ and $f(2) = 0$. If $n \geq 3$, $f(n-2) = 0$ and $f(n-1) = 0$, then (E1) implies that $(n^2-1)f(n) = 0$, which implies that $f(n) = 0$ as well.