

**11209.** Proposed by Alexander Dubinov and Irina Dubinova, Russian Federal Nuclear Center, Sarov, Russia. Consider the following system of  $n$  equations in  $n$  positive unknowns  $x_1, \dots, x_n$  and  $n$  positive given numbers  $a_1, \dots, a_n$ :

$$\left( \sum_{k=1}^n x_k \right)^{x_j} = a_j \quad (1 \leq j \leq n).$$

Find a formula in closed form for  $x_1, \dots, x_n$  in terms of  $a_1, \dots, a_n$ .

*Solution by Christopher Carl Heckman, Arizona State Univeristy, Tempe, AZ:* If  $a_j \neq 1$ , for all  $j$ , then a closed form is

$$x_j = \frac{e^{W(\ln(a_1 \cdots a_n))}}{\ln(a_1 \cdots a_n)} \cdot \ln a_j, \quad (1 \leq j \leq n), \quad (1a)$$

provided  $a_1 \cdots a_n \geq e^{-1/e}$ ; otherwise there is no solution. Here,  $W$  is the Lambert-W function: the inverse of the function  $f(x) = xe^x$  over the interval  $[-1, \infty)$ . If some  $a_i = 1$ , but not every  $a_j$  is 1, there are no solutions. Otherwise, if  $a_j = 1$  for all  $j$ , it will be shown that the only extra requirement on the variables is that

$$\sum_{k=1}^n x_k = 1. \quad (1b)$$

First, note that

$$(a_1)^{1/x_1} = \sum_{k=1}^n x_k = (a_j)^{1/x_j},$$

for any  $j$ , so that  $\frac{1}{x_j} \ln a_j = \frac{1}{x_1} \ln a_1$ . Thus if  $a_i = 1$ ,  $a_1 = 1$ ; and consequently  $a_j = 1$  follows, for all  $j$ . If this does not contradict the actual values of the constants, the system collapses to the single equation given in (1b), since  $1^{1/x_j} = 1$ .

If  $a_j \neq 1$  for all  $j$ , we can safely write

$$x_j = \frac{\ln a_j}{\ln a_1} x_1. \quad (2)$$

Thus

$$\sum_{k=1}^n x_k = x_1 \sum_{k=1}^n \frac{\ln a_k}{\ln a_1} = \frac{\ln(a_1 \cdots a_n)}{\ln a_1} x_1;$$

combining this equation with one of the original equations yields

$$(Ax_1)^{x_1} = a_1, \quad (3)$$

where  $A = \frac{\ln(a_1 \cdots a_n)}{\ln a_1} = \log_{a_1}(a_1 \cdots a_n)$ .

Now we need to solve (3), and this is where the Lambert-W function comes in. After we raise both sides of (3) to the  $A$ th power, we obtain the equation

$$(Ax_1)^{Ax_1} = (a_1)^A = a_1 \cdots a_n,$$

which becomes  $y^y = a_1 \cdots a_n$  if we let  $y = Ax_1$ . Now we let  $y = e^z$  and obtain the equation

$$e^{ze^z} = (e^z)^{(e^z)} = y^y = a_1 \cdots a_n.$$

Taking the natural log of both sides of this equation yields

$$ze^z = \ln(a_1 \cdots a_n),$$

so that  $z = W(\ln(a_1 \cdots a_n))$ . (Note that  $z$  exists, provided  $a_1 \cdots a_n > e^{g(-1)} = e^{-1/e}$ , where  $g(x) = xe^x$ .)

Now we backtrack, noting that

$$x_1 = \frac{y}{A} = \frac{e^z}{A} = \frac{e^{W(\ln(a_1 \cdots a_n))} \cdot \ln a_1}{\ln(a_1 \cdots a_n)}. \quad (4)$$

Now, if we combine (2) with (4), we obtain equation (1a), for all  $j$ .